Lectures on Differential Geometry of Modules and Rings

G. Sardanashvily

Department of Theoretical Physics, Moscow State University, Moscow, Russia

Abstract

Generalizing differential geometry of smooth vector bundles formulated in algebraic terms of the ring of smooth functions, its derivations and the Koszul connection, one can define differential operators, differential calculus and connections on modules over arbitrary commutative, graded commutative and noncommutative rings. For instance, this is the case of quantum theory, SUSY theory and noncommutative geometry, respectively. The relevant material on this subject is summarized.

Contents

Introduction **3**

1 Commutative geometry **5**

1.1 Commutative algebra . . . 5
1.2 Differential operators on modules and rings . . . 10
1.3 Connections on modules and rings . . . 13
1.4 Differential calculus over a commutative ring . . . 17
1.5 Local-ringed spaces . . . 19
1.6 Differential geometry of $C^\infty(X)$-modules . . . 21
1.7 Connections on local-ringed spaces . . . 25

2 Geometry of quantum systems **29**

2.1 Geometry of Banach manifolds . . . 29
2.2 Geometry of Hilbert manifolds . . . 34
2.3 Hilbert and C^*-algebra bundles . . . 40
2.4 Connections on Hilbert and C^*-algebra bundles . . . 44
2.5 Instantwise quantization . . . 45
2.6 Berry connection . . . 48

3 Supergeometry **53**

3.1 Graded tensor calculus . . . 53
3.2 Graded differential calculus and connections . . . 56
3.3 Geometry of graded manifolds . . . 62
3.4 Supermanifolds . . . 69
3.5 Supervector bundles . . . 74
3.6 Superconnections . . . 77

4 Non-commutative geometry **81**

4.1 Modules over C^*-algebras . . . 81
4.2 Non-commutative differential calculus . . . 84
4.3 Differential operators in non-commutative geometry . . . 87
4.4 Connections in non-commutative geometry . . . 92
4.5 Matrix geometry . . . 95
4.6 Connes' non-commutative geometry . . . 97
4.7 Differential calculus over Hopf algebras . . . 100

5 Appendix. Cohomology **111**
5.1 Cohomology of complexes . . . 111
5.2 Cohomology of Lie algebras . . . 114
5.3 Sheaf cohomology . . . 116

Bibliography **125**

Index **131**

Introduction

Geometry of classical mechanics and field theory is mainly differential geometry of finite-dimensional smooth manifolds and fibre bundles [42, 60, 79].

At the same time, the standard mathematical language of quantum mechanics and field theory has been long far from geometry. In the last twenty years, the incremental development of new physical ideas in quantum theory (including super- and BRST symmetries, geometric and deformation quantization, topological field theory, anomalies, non-commutativity, strings and branes) has called into play advanced geometric techniques, based on the deep interplay between algebra, geometry and topology [41, 76].

Geometry in quantum systems speaks mainly the algebraic language of rings, modules and sheaves due to the fact that the basic ingredients in the differential calculus and differential geometry of smooth manifolds (except non-linear differential operators) can be restarted in a pure algebraic way.

Let X be a smooth manifold and $C^\infty(X)$ the ring of smooth real functions on X. A key point is that, by virtue of the well-known Serre–Swan theorem, a $C^\infty(X)$-module is finitely generated projective iff it is isomorphic to the module of sections of some smooth vector bundle over X. Moreover, this isomorphism is a categorial equivalence. Therefore, differential geometry of smooth vector bundles can be adequately formulated in algebraic terms of the ring $C^\infty(X)$, its derivations and the Koszul connection (Section 1.6).

In a general setting, let $\mathcal{K}$ be a commutative ring, $\mathcal{A}$ an arbitrary commutative $\mathcal{K}$-ring, and P, Q some $\mathcal{A}$-modules. The $\mathcal{K}$-linear Q-valued differential operators on P can be defined [44, 51]. The representative objects of the functors $Q \to \mathrm{Diff}_s(P,Q)$ are the jet modules $\mathcal{J}^sP$ of P. Using the first order jet module $\mathcal{J}^1P$, one also restarts the notion of a connection on an $\mathcal{A}$-module P [50, 61]. Such a connection assigns to each derivation $\tau \in \mathfrak{d}\mathcal{A}$ of a $\mathcal{K}$-ring $\mathcal{A}$ a first order P-valued differential operator ∇_τ on P obeying the Leibniz rule

$$\nabla_\tau(ap) = \tau(a)p + a\nabla_\tau(p).$$

Similarly, connections on local-ringed spaces are introduced (Section 1.7).

As was mentioned above, if P is a $C^\infty(X)$-module of sections of a smooth vector bundle $Y \to X$, we come to the familiar notions of a linear differential operator on Y, the jets of sections of $Y \to X$ and a linear connection on $Y \to X$.

In quantum theory, Banach and Hilbert manifolds, Hilbert bundles and bundles of C^*-algebras over smooth manifolds are considered. Their differential geometry also is formulated as geometry of modules, in particular, $C^\infty(X)$-modules (Chapter 2).

Let $\mathcal{K}$ be a commutative ring, $\mathcal{A}$ a (commutative or non-commutative) $\mathcal{K}$-ring, and $\mathcal{Z}(\mathcal{A})$ the center of $\mathcal{A}$. Derivations of $\mathcal{A}$ make up a Lie $\mathcal{K}$-algebra $\mathfrak{d}\mathcal{A}$. Let us consider the Chevalley–Eilenberg complex of $\mathcal{K}$-multilinear morphisms of $\mathfrak{d}\mathcal{A}$ to $\mathcal{A}$, seen as a $\mathfrak{d}\mathcal{A}$-module [39, 61]. Its subcomplex $\mathcal{O}^*(\mathfrak{d}\mathcal{A}, d)$ of $\mathcal{Z}(\mathcal{A})$-multilinear morphisms is a differential graded algebra, called the Chevalley–Eilenberg differential calculus over $\mathcal{A}$. It contains the minimal differential calculus $\mathcal{O}^*\mathcal{A}$ generated by elements da, $a \in \mathcal{A}$. If $\mathcal{A}$ is the $\mathbb{R}$-ring $C^\infty(X)$ of smooth real functions on a smooth manifold X, the module $\mathfrak{d}C^\infty(X)$ of its derivations is the Lie algebra of vector fields on X and the Chevalley–Eilenberg differential calculus over $C^\infty(X)$ is exactly the algebra of exterior forms on a manifold X where the Chevalley–Eilenberg coboundary operator d coincides with the exterior differential, i.e., $\mathcal{O}^*(\mathfrak{d}C^\infty(X), d)$ is the familiar de Rham complex. In a general setting, one therefore can think of elements of the Chevalley–Eilenberg differential calculus $\mathcal{O}^k(\mathfrak{d}\mathcal{A}, d)$ over an algebra $\mathcal{A}$ as being differential forms over $\mathcal{A}$.

Similarly, the Chevalley–Eilenberg differential calculus over a graded commutative ring is constructed [39, 42, 80]. This is the case of SUSY theory and supergeometry (Chapter 3). In supergeometry, connections on graded manifolds and supervector bundles are defined as those on graded modules over a graded commutative ring and graded local-ringed spaces are defined [5, 61, 80].

Non-commutative geometry also is developed as a generalization of the calculus in commutative rings of smooth functions [23, 53]. In a general setting, any non-commutative $\mathcal{K}$-ring $\mathcal{A}$ over a commutative ring $\mathcal{K}$ can be called into play. One can consider the above mentioned Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}$ over $\mathcal{A}$, differential operators and connections on $\mathcal{A}$-modules (but not their jets) (Chapter 4). If the derivation $\mathcal{K}$-module $\mathfrak{d}\mathcal{A}$ is a finite projective module with respect to the center of $\mathcal{A}$, one can treat the triple $(\mathcal{A}, \mathfrak{d}\mathcal{A}, \mathcal{O}^*\mathcal{A})$ as a non-commutative space.

Note that, in non-commutative geometry, different definitions of a differential operator on modules over a non-commutative ring have been suggested [9, 34, 56], but there is no a satisfactory construction of higher order differential operator [41, 77] Roughly speaking, the difficulty lies in the fact that, if ∂ is a derivation of a non-commutative $\mathcal{K}$-ring $\mathcal{A}$, the product $a\partial$, $a \in \mathcal{A}$, need not be so. There are also different definitions of a connection on modules over a non-commutative ring [35, 53, 61].

Chapter 1

Commutative geometry

We start with differential calculus and differential geometry of modules over commutative rings. In particular, this is the case of differential geometry of smooth manifolds and smooth vector bundles (Section 1.6).

1.1 Commutative algebra

In this Section, the relevant basics on modules over commutative algebras is summarized [54, 57].

An *algebra* $\mathcal{A}$ is an additive group which is additionally provided with distributive multiplication. All algebras throughout the Lectures are associative, unless they are Lie algebras. A *ring* is a *unital* algebra, i.e., it contains a unit element $\mathbf{1}$. Unless otherwise stated, we assume that $\mathbf{1} \neq 0$, i.e., a ring does not reduce to the zero element. One says that A is a *division algebra* if it has no a divisor of zero, i.e., $ab = 0$, $a, b \in A$, implies either $a = 0$ or $b = 0$. Non-zero elements of a ring form a multiplicative monoid. If this multiplicative monoid is a multiplicative group, one says that the ring has a multiplicative inverse. A ring A has a multiplicative inverse iff it is a division algebra. A *field* is a commutative ring whose non-zero elements make up a multiplicative group.

A subset $\mathcal{I}$ of an algebra $\mathcal{A}$ is called a left (resp. right) *ideal* if it is a subgroup of the additive group $\mathcal{A}$ and $ab \in \mathcal{I}$ (resp. $ba \in \mathcal{I}$) for all $a \in \mathcal{A}$, $b \in \mathcal{I}$. If $\mathcal{I}$ is both a left and right ideal, it is called a two-sided ideal. An ideal is a subalgebra, but a *proper* ideal (i.e., $\mathcal{I} \neq \mathcal{A}$) of a ring is not a subring because it does not contain a unit element.

Let $\mathcal{A}$ be a commutative ring. Of course, its ideals are two-sided. Its proper ideal is said to be *maximal* if it does not belong to another proper ideal. A commutative ring $\mathcal{A}$ is called *local* if it has a unique maximal ideal. This ideal consists of all non-invertible elements of $\mathcal{A}$. A proper two-sided ideal $\mathcal{I}$ of a commutative ring is called *prime* if $ab \in \mathcal{I}$ implies either $a \in \mathcal{I}$ or $b \in \mathcal{I}$. Any maximal two-sided ideal is prime. Given a two-sided ideal $\mathcal{I} \subset \mathcal{A}$, the additive factor group $\mathcal{A}/\mathcal{I}$ is an algebra, called the *factor algebra*. If $\mathcal{A}$

is a ring, then $\mathcal{A}/\mathcal{I}$ is so. If $\mathcal{I}$ is a prime ideal, the factor ring $\mathcal{A}/\mathcal{I}$ has no divisor of zero, and it is a field if $\mathcal{I}$ is a maximal ideal.

Given an algebra $\mathcal{A}$, an additive group P is said to be a left (resp. right) $\mathcal{A}$-*module* if it is provided with distributive multiplication $\mathcal{A} \times P \to P$ by elements of $\mathcal{A}$ such that $(ab)p = a(bp)$ (resp. $(ab)p = b(ap)$) for all $a, b \in \mathcal{A}$ and $p \in P$. If $\mathcal{A}$ is a ring, one additionally assumes that $\mathbf{1}p = p = p\mathbf{1}$ for all $p \in P$. Left and right module structures are usually written by means of left and right multiplications $(a, p) \mapsto ap$ and $(a, p) \mapsto pa$, respectively. If P is both a left module over an algebra $\mathcal{A}$ and a right module over an algebra $\mathcal{A}'$, it is called an $(\mathcal{A} - \mathcal{A}')$-bimodule (an $\mathcal{A}$-bimodule if $\mathcal{A} = \mathcal{A}'$). If $\mathcal{A}$ is a commutative algebra, an $(\mathcal{A} - \mathcal{A})$-bimodule P is said to be *commutative* if $ap = pa$ for all $a \in \mathcal{A}$ and $p \in P$. Any left or right module over a commutative algebra $\mathcal{A}$ can be brought into a commutative bimodule. Therefore, unless otherwise stated, any module over a commutative algebra $\mathcal{A}$ is called an $\mathcal{A}$-module.

A module over a field is called a *vector space*. If an algebra $\mathcal{A}$ is a module over a ring $\mathcal{K}$, it is said to be a $\mathcal{K}$-*algebra*. Any algebra can be seen as a $\mathbb{Z}$-algebra.

Remark 1.1.1: Any $\mathcal{K}$-algebra $\mathcal{A}$ can be extended to a unital algebra $\widetilde{\mathcal{A}}$ by the adjunction of the identity $\mathbf{1}$ to $\mathcal{A}$. The algebra $\widetilde{\mathcal{A}}$, called the *unital extension* of $\mathcal{A}$, is defined as the direct sum of $\mathcal{K}$-modules $\mathcal{K} \oplus \mathcal{A}$ provided with the multiplication

$$(\lambda_1, a_1)(\lambda_2, a_2) = (\lambda_1\lambda_2, \lambda_1 a_2 + \lambda_2 a_1 + a_1 a_2), \quad \lambda_1, \lambda_2 \in \mathcal{K}, \quad a_1, a_2 \in \mathcal{A}.$$

Elements of $\widetilde{\mathcal{A}}$ can be written as $(\lambda, a) = \lambda\mathbf{1} + a$, $\lambda \in \mathcal{K}$, $a \in \mathcal{A}$.

Let us note that, if $\mathcal{A}$ is a unital algebra, the identity $\mathbf{1}_{\mathcal{A}}$ in $\mathcal{A}$ fails to be that in $\widetilde{\mathcal{A}}$. In this case, the algebra $\widetilde{\mathcal{A}}$ is isomorphic to the product of $\mathcal{A}$ and the algebra $\mathcal{K}(\mathbf{1} - \mathbf{1}_{\mathcal{A}})$. $\diamondsuit$

In this Chapter, all associative algebras are assumed to be commutative, unless they are graded.

The following are standard constructions of new modules from old ones.

• The *direct sum* $P_1 \oplus P_2$ of $\mathcal{A}$-modules P_1 and P_2 is the additive group $P_1 \times P_2$ provided with the $\mathcal{A}$-module structure

$$a(p_1, p_2) = (ap_1, ap_2), \qquad p_{1,2} \in P_{1,2}, \qquad a \in \mathcal{A}.$$

Let $\{P_i\}_{i \in I}$ be a set of modules. Their direct sum $\oplus P_i$ consists of elements $(\ldots, p_i, \ldots)$ of the Cartesian product $\prod P_i$ such that $p_i \neq 0$ at most for a finite number of indices $i \in I$.

• The *tensor product* $P \otimes Q$ of $\mathcal{A}$-modules P and Q is an additive group which is generated by elements $p \otimes q$, $p \in P$, $q \in Q$, obeying the relations

$$(p + p') \otimes q = p \otimes q + p' \otimes q, \quad p \otimes (q + q') = p \otimes q + p \otimes q',$$
$$pa \otimes q = p \otimes aq, \qquad p \in P, \qquad q \in Q, \qquad a \in \mathcal{A},$$

and it is provided with the $\mathcal{A}$-module structure

$$a(p \otimes q) = (ap) \otimes q = p \otimes (qa) = (p \otimes q)a.$$

If the ring $\mathcal{A}$ is treated as an $\mathcal{A}$-module, the tensor product $\mathcal{A}\otimes_{\mathcal{A}} Q$ is canonically isomorphic to Q via the assignment

$$\mathcal{A}\otimes_{\mathcal{A}} Q \ni a\otimes q \leftrightarrow aq \in Q.$$

• Given a submodule Q of an $\mathcal{A}$-module P, the quotient P/Q of the additive group P with respect to its subgroup Q is also provided with an $\mathcal{A}$-module structure. It is called a *factor module*.

• The set $\mathrm{Hom}_{\mathcal{A}}(P,Q)$ of $\mathcal{A}$-linear morphisms of an $\mathcal{A}$-module P to an $\mathcal{A}$-module Q is naturally an $\mathcal{A}$-module. The $\mathcal{A}$-module $P^* = \mathrm{Hom}_{\mathcal{A}}(P,\mathcal{A})$ is called the *dual* of an $\mathcal{A}$-module P. There is a natural monomorphism $P\to P^{**}$.

An $\mathcal{A}$-module P is called *free* if it has a *basis*, i.e., a linearly independent subset $I\subset P$ spanning P such that each element of P has a unique expression as a linear combination of elements of I with a finite number of non-zero coefficients from an algebra $\mathcal{A}$. Any vector space is free. Any module is isomorphic to a quotient of a free module. A module is said to be *finitely generated* (or of *finite rank*) if it is a quotient of a free module with a finite basis.

One says that a module P is *projective* if it is a direct summand of a free module, i.e., there exists a module Q such that $P\oplus Q$ is a free module. A module P is projective iff $P=\mathbf{p}S$ where S is a free module and $\mathbf{p}$ is a projector of S, i.e., $\mathbf{p}^2=\mathbf{p}$. If P is a projective module of finite rank over a ring, then its dual P^* is so, and P^{**} is isomorphic to P.

THEOREM 1.1.1: Any projective module over a local ring is free. □

Now we focus on exact sequences, direct and inverse limits of modules [57, 62].

A composition of module morphisms

$$P \xrightarrow{i} Q \xrightarrow{j} T$$

is said to be *exact* at Q if $\mathrm{Ker}\, j = \mathrm{Im}\, i$. A composition of module morphisms

$$0\to P \xrightarrow{i} Q \xrightarrow{j} T \to 0 \tag{1.1.1}$$

is called a *short exact sequence* if it is exact at all the terms P, Q, and T. This condition implies that: (i) i is a monomorphism, (ii) $\mathrm{Ker}\, j=\mathrm{Im}\, i$, and (iii) j is an epimorphism onto the quotient $T=Q/P$.

THEOREM 1.1.2: Given an exact sequence of modules (1.1.1) and another $\mathcal{A}$-module R, the sequence of modules

$$0\to \mathrm{Hom}_{\mathcal{A}}(T,R) \xrightarrow{j^*} \mathrm{Hom}_{\mathcal{A}}(Q,R) \xrightarrow{i^*} \mathrm{Hom}\,(P,R) \tag{1.1.2}$$

is exact at the first and second terms, i.e., j^* is a monomorphism, but i^* need not be an epimorphism. □

One says that the exact sequence (1.1.1) is *split* if there exists a monomorphism $s : T \to Q$ such that $j \circ s = \mathrm{Id}\, T$ or, equivalently,

$$Q = i(P) \oplus s(T) \cong P \oplus T.$$

The exact sequence (1.1.1) is always split if T is a projective module.

A *directed set* I is a set with an order relation $<$ which satisfies the following three conditions:

(i) $i < i$, for all $i \in I$;

(ii) if $i < j$ and $j < k$, then $i < k$;

(iii) for any $i, j \in I$, there exists $k \in I$ such that $i < k$ and $j < k$.

It may happen that $i \neq j$, but $i < j$ and $j < i$ simultaneously.

A family of modules $\{P_i\}_{i \in I}$ (over the same algebra), indexed by a directed set I, is called a *direct system* if, for any pair $i < j$, there exists a morphism $r^i_j : P_i \to P_j$ such that

$$r^i_i = \mathrm{Id}\, P_i, \qquad r^i_j \circ r^j_k = r^i_k, \qquad i < j < k.$$

A direct system of modules admits a *direct limit*. This is a module P_∞ together with morphisms $r^i_\infty : P_i \to P_\infty$ such that $r^i_\infty = r^j_\infty \circ r^i_j$ for all $i < j$. The module P_∞ consists of elements of the direct sum $\bigoplus\limits_I P_i$ modulo the identification of elements of P_i with their images in P_j for all $i < j$. An example of a direct system is a *direct sequence*

$$P_0 \longrightarrow P_1 \longrightarrow \cdots P_i \xrightarrow{r^i_{i+1}} \cdots, \qquad I = \mathbb{N}. \tag{1.1.3}$$

It should be noted that direct limits also exist in the categories of commutative algebras and rings, but not in categories whose objects are non-Abelian groups.

THEOREM 1.1.3: Direct limits commute with direct sums and tensor products of modules. Namely, let $\{P_i\}$ and $\{Q_i\}$ be two direct systems of modules over the same algebra which are indexed by the same directed set I, and let P_∞ and Q_∞ be their direct limits. Then the direct limits of the direct systems $\{P_i \oplus Q_i\}$ and $\{P_i \otimes Q_i\}$ are $P_\infty \oplus Q_\infty$ and $P_\infty \otimes Q_\infty$, respectively. □

A morphism of a direct system $\{P_i, r^i_j\}_I$ to a direct system $\{Q_{i'}, \rho^{i'}_{j'}\}_{I'}$ consists of an order preserving map $f : I \to I'$ and morphisms $F_i : P_i \to Q_{f(i)}$ which obey the compatibility conditions

$$\rho^{f(i)}_{f(j)} \circ F_i = F_j \circ r^i_j.$$

If P_∞ and Q_∞ are limits of these direct systems, there exists a unique morphism $F_\infty : P_\infty \to Q_\infty$ such that

$$\rho^{f(i)}_\infty \circ F_i = F_\infty \circ r^i_\infty.$$

Moreover, direct limits preserve monomorphisms and epimorphisms. To be precise, if all $F_i : P_i \to Q_{f(i)}$ are monomorphisms or epimorphisms, so is $\Phi_\infty : P_\infty \to Q_\infty$. As a consequence, the following holds.

THEOREM 1.1.4: Let short exact sequences

$$0 \to P_i \xrightarrow{F_i} Q_i \xrightarrow{\Phi_i} T_i \to 0 \tag{1.1.4}$$

for all $i \in I$ define a short exact sequence of direct systems of modules $\{P_i\}_I$, $\{Q_i\}_I$, and $\{T_i\}_I$ which are indexed by the same directed set I. Then there exists a short exact sequence of their direct limits

$$0 \to P_\infty \xrightarrow{F_\infty} Q_\infty \xrightarrow{\Phi_\infty} T_\infty \to 0. \tag{1.1.5}$$

□

In particular, the direct limit of factor modules Q_i/P_i is the factor module Q_∞/P_∞. By virtue of Theorem 1.1.3, if all the exact sequences (1.1.4) are split, the exact sequence (1.1.5) is well.

Example 1.1.2: Let P be an $\mathcal{A}$-module. We denote $P^{\otimes k} = \overset{k}{\otimes} P$. Let us consider the direct system of $\mathcal{A}$-modules with respect to monomorphisms

$$\mathcal{A} \longrightarrow (\mathcal{A} \oplus P) \longrightarrow \cdots (\mathcal{A} \oplus P \oplus \cdots \oplus P^{\otimes k}) \longrightarrow \cdots.$$

Its direct limit

$$\otimes P = \mathcal{A} \oplus P \oplus \cdots \oplus P^{\otimes k} \oplus \cdots \tag{1.1.6}$$

is an $\mathbb{N}$-graded $\mathcal{A}$-algebra with respect to the tensor product $\otimes$. It is called the *tensor algebra* of a module P. Its quotient with respect to the ideal generated by elements $p \otimes p' + p' \otimes p$, $p, p' \in P$, is an $\mathbb{N}$-graded commutative algebra, called the *exterior algebra* of a module P. ♢

We restrict our consideration of inverse systems of modules to *inverse sequences*

$$P^0 \longleftarrow P^1 \longleftarrow \cdots P^i \overset{\pi_i^{i+1}}{\longleftarrow} \cdots. \tag{1.1.7}$$

Its *inductive limit* (*the inverse limit*) is a module P^∞ together with morphisms $\pi_i^\infty : P^\infty \to P^i$ such that $\pi_i^\infty = \pi_i^j \circ \pi_j^\infty$ for all $i < j$. It consists of elements $(\ldots, p^i, \ldots)$, $p^i \in P^i$, of the Cartesian product $\prod P^i$ such that $p^i = \pi_i^j(p^j)$ for all $i < j$.

THEOREM 1.1.5: Inductive limits preserve monomorphisms, but not epimorphisms. If a sequence

$$0 \to P^i \xrightarrow{F^i} Q^i \xrightarrow{\Phi^i} T^i, \qquad i \in \mathbb{N},$$

of inverse systems of modules $\{P^i\}$, $\{Q^i\}$ and $\{T^i\}$ is exact, so is the sequence of the inductive limits

$$0 \to P^\infty \xrightarrow{F^\infty} Q^\infty \xrightarrow{\Phi^\infty} T^\infty.$$

$\square$

In contrast with direct limits, the inductive ones exist in the category of groups which are not necessarily commutative.

Example 1.1.3: Let $\{P_i\}$ be a direct sequence of modules. Given another module Q, the modules $\mathrm{Hom}\,(P_i, Q)$ make up an inverse system such that its inductive limit is isomorphic to $\mathrm{Hom}\,(P_\infty, Q)$. $\diamondsuit$

1.2 Differential operators on modules and rings

This Section addresses the notion of a (linear) differential operator on a module over a commutative ring [44, 51, 42].

Let $\mathcal{K}$ be a commutative ring and $\mathcal{A}$ a commutative $\mathcal{K}$-ring. Let P and Q be $\mathcal{A}$-modules. The $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ of $\mathcal{K}$-module homomorphisms $\Phi : P \to Q$ can be endowed with the two different $\mathcal{A}$-module structures

$$(a\Phi)(p) = a\Phi(p), \qquad (\Phi \bullet a)(p) = \Phi(ap), \qquad a \in \mathcal{A}, \quad p \in P. \tag{1.2.1}$$

For the sake of convenience, we will refer to the second one as the $\mathcal{A}^\bullet$-module structure. Let us put

$$\delta_a \Phi = a\Phi - \Phi \bullet a, \qquad a \in \mathcal{A}. \tag{1.2.2}$$

Definition 1.2.1: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is called a Q-valued *differential operator* of order s on P if

$$\delta_{a_0} \circ \cdots \circ \delta_{a_s} \Delta = 0$$

for any tuple of $s+1$ elements $a_0, \ldots, a_s$ of $\mathcal{A}$. $\square$

The set $\mathrm{Diff}_s(P,Q)$ of these operators inherits the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (1.2.1). Of course, an s-order differential operator is also of $(s+1)$-order.

In particular, zero order differential operators obey the condition

$$\delta_a \Delta(p) = a\Delta(p) - \Delta(ap) = 0, \qquad a \in \mathcal{A}, \qquad p \in P,$$

and, consequently, they coincide with $\mathcal{A}$-module morphisms $P \to Q$. A first order differential operator Δ satisfies the condition

$$\delta_b \circ \delta_a \Delta(p) = ba\Delta(p) - b\Delta(ap) - a\Delta(bp) + \Delta(abp) = 0, \quad a, b \in \mathcal{A}. \tag{1.2.3}$$

The following fact reduces the study of Q-valued differential operators on an $\mathcal{A}$-module P to that of Q-valued differential operators on the ring $\mathcal{A}$.

PROPOSITION 1.2.2: Let us consider the $\mathcal{A}$-module morphism

$$h_s : \mathrm{Diff}_s(\mathcal{A}, Q) \to Q, \qquad h_s(\Delta) = \Delta(\mathbf{1}). \tag{1.2.4}$$

Any Q-valued s-order differential operator $\Delta \in \mathrm{Diff}_s(P, Q)$ on P uniquely factorizes

$$\Delta : P \xrightarrow{\mathfrak{f}_\Delta} \mathrm{Diff}_s(\mathcal{A}, Q) \xrightarrow{h_s} Q \tag{1.2.5}$$

through the morphism h_s (1.2.4) and some homomorphism

$$\mathfrak{f}_\Delta : P \to \mathrm{Diff}_s(\mathcal{A}, Q), \qquad (\mathfrak{f}_\Delta p)(a) = \Delta(ap), \qquad a \in \mathcal{A}, \tag{1.2.6}$$

of the $\mathcal{A}$-module P to the $\mathcal{A}^\bullet$-module $\mathrm{Diff}_s(\mathcal{A}, Q)$ [51]. The assignment $\Delta \mapsto \mathfrak{f}_\Delta$ defines the isomorphism

$$\mathrm{Diff}_s(P, Q) = \mathrm{Hom}_{\mathcal{A}-\mathcal{A}^\bullet}(P, \mathrm{Diff}_s(\mathcal{A}, Q)). \tag{1.2.7}$$

□

Let $P = \mathcal{A}$. Any zero order Q-valued differential operator Δ on $\mathcal{A}$ is defined by its value $\Delta(\mathbf{1})$. Then there is an isomorphism $\mathrm{Diff}_0(\mathcal{A}, Q) = Q$ via the association

$$Q \ni q \mapsto \Delta_q \in \mathrm{Diff}_0(\mathcal{A}, Q),$$

where Δ_q is given by the equality $\Delta_q(\mathbf{1}) = q$. A first order Q-valued differential operator Δ on $\mathcal{A}$ fulfils the condition

$$\Delta(ab) = b\Delta(a) + a\Delta(b) - ba\Delta(\mathbf{1}), \qquad a, b \in \mathcal{A}.$$

It is called a Q-valued *derivation* of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$, i.e., the *Leibniz rule*

$$\Delta(ab) = \Delta(a)b + a\Delta(b), \qquad a, b \in \mathcal{A}, \tag{1.2.8}$$

holds. One obtains at once that any first order differential operator on $\mathcal{A}$ falls into the sum

$$\Delta(a) = a\Delta(\mathbf{1}) + [\Delta(a) - a\Delta(\mathbf{1})]$$

of the zero order differential operator $a\Delta(\mathbf{1})$ and the derivation $\Delta(a) - a\Delta(\mathbf{1})$. If ∂ is a derivation of $\mathcal{A}$, then $a\partial$ is well for any $a \in \mathcal{A}$. Hence, derivations of $\mathcal{A}$ constitute an $\mathcal{A}$-module $\mathfrak{d}(\mathcal{A}, Q)$, called the *derivation module*. There is the $\mathcal{A}$-module decomposition

$$\mathrm{Diff}_1(\mathcal{A}, Q) = Q \oplus \mathfrak{d}(\mathcal{A}, Q). \tag{1.2.9}$$

Remark 1.2.1: Let us recall that, given a (non-commutative) $\mathcal{K}$-algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule Q, by a Q-valued *derivation* of $\mathcal{A}$ is meant a $\mathcal{K}$-module morphism $u : \mathcal{A} \to Q$ which obeys the *Leibniz rule*

$$u(ab) = u(a)b + au(b), \qquad a, b \in \mathcal{A}. \tag{1.2.10}$$

It should be emphasized that this derivation rule differs from that (3.2.3) of graded derivations. A Q-valued derivation u of A is called *inner* if there exists an element $q \in Q$ such that $u(a) = qa - aq$. $\diamondsuit$

If $Q = \mathcal{A}$, the derivation module $\mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ is also a Lie algebra over the ring $\mathcal{K}$ with respect to the Lie bracket

$$[u, u'] = u \circ u' - u' \circ u, \qquad u, u' \in \mathcal{A}. \tag{1.2.11}$$

Accordingly, the decomposition (1.2.9) takes the form

$$\mathrm{Diff}_1(\mathcal{A}) = \mathcal{A} \oplus \mathfrak{d}\mathcal{A}. \tag{1.2.12}$$

An s-order differential operator on a module P is represented by a zero order differential operator on the module of s-order jets of P as follows.

Given an $\mathcal{A}$-module P, let us consider the tensor product $\mathcal{A} \otimes_{\mathcal{K}} P$ of $\mathcal{K}$-modules $\mathcal{A}$ and P. We put

$$\delta^b(a \otimes p) = (ba) \otimes p - a \otimes (bp), \qquad p \in P, \qquad a, b \in \mathcal{A}. \tag{1.2.13}$$

Let us denote by μ^{k+1} the submodule of $\mathcal{A} \otimes_{\mathcal{K}} P$ generated by elements of the type

$$\delta^{b_0} \circ \cdots \circ \delta^{b_k}(a \otimes p).$$

The *k-order jet module* $\mathcal{J}^k(P)$ of a module P is defined as the quotient of the $\mathcal{K}$-module $\mathcal{A} \otimes_{\mathcal{K}} P$ by μ^{k+1}. We denote its elements $c \otimes_k p$.

In particular, the first order jet module $\mathcal{J}^1(P)$ consists of elements $c \otimes_1 p$ modulo the relations

$$\delta^a \circ \delta^b(\mathbf{1} \otimes_1 p) = ab \otimes_1 p - b \otimes_1 (ap) - a \otimes_1 (bp) + \mathbf{1} \otimes_1 (abp) = 0. \tag{1.2.14}$$

The $\mathcal{K}$-module $\mathcal{J}^k(P)$ is endowed with the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures

$$b(a \otimes_k p) = ba \otimes_k p, \qquad b \bullet (a \otimes_k p) = a \otimes_k (bp). \tag{1.2.15}$$

There exists the module morphism

$$J^k : P \ni p \mapsto \mathbf{1} \otimes_k p \in \mathcal{J}^k(P) \tag{1.2.16}$$

of the $\mathcal{A}$-module P to the $\mathcal{A}^\bullet$-module $\mathcal{J}^k(P)$ such that $\mathcal{J}^k(P)$, seen as an $\mathcal{A}$-module, is generated by elements $J^k p$, $p \in P$.

Due to the natural monomorphisms $\mu^r \to \mu^s$ for all $r > s$, there are $\mathcal{A}$-module epimorphisms of jet modules

$$\pi^{i+1}_i : \mathcal{J}^{i+1}(P) \to \mathcal{J}^i(P). \tag{1.2.17}$$

In particular,

$$\pi^1_0 : \mathcal{J}^1(P) \ni a \otimes_1 p \mapsto ap \in P. \tag{1.2.18}$$

The above mentioned relation between differential operators on modules and jets of modules is stated by the following theorem [51].

THEOREM 1.2.3: Any Q-valued differential operator Δ of order k on an $\mathcal{A}$-module P factorizes uniquely

$$\Delta : P \xrightarrow{J^k} \mathcal{J}^k(P) \longrightarrow Q$$

through the morphism J^k (1.2.16) and some $\mathcal{A}$-module homomorphism $\mathfrak{f}^\Delta : \mathcal{J}^k(P) \to Q$. □

The proof is based on the fact that the morphism J^k (1.2.16) is a k-order $\mathcal{J}^k(P)$-valued differential operator on P. Let us denote

$$J : P \ni p \mapsto \mathbf{1} \otimes p \in \mathcal{A} \otimes P.$$

Then, for any $\mathfrak{f} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes P, Q)$, we obtain

$$\delta_b(\mathfrak{f} \circ J)(p) = \mathfrak{f}(\delta^b(\mathbf{1} \otimes p)).$$

The correspondence $\Delta \mapsto \mathfrak{f}^\Delta$ defines an $\mathcal{A}$-module isomorphism

$$\mathrm{Diff}_s(P, Q) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^s(P), Q). \tag{1.2.19}$$

1.3 Connections on modules and rings

We employ the jets of modules in previous Section in order to introduce connections on modules and commutative rings [61].

Let us consider the jet modules $\mathcal{J}^s = \mathcal{J}^s(\mathcal{A})$ of the ring $\mathcal{A}$ itself. In particular, the first order jet module $\mathcal{J}^1$ consists of the elements $a \otimes_1 b$, $a, b \in \mathcal{A}$, subject to the relations

$$ab \otimes_1 \mathbf{1} - b \otimes_1 a - a \otimes_1 b + \mathbf{1} \otimes_1 (ab) = 0. \tag{1.3.1}$$

The $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (1.2.15) on $\mathcal{J}^1$ read

$$c(a \otimes_1 b) = (ca) \otimes_1 b, \qquad c \bullet (a \otimes_1 b) = a \otimes_1 (cb) = (a \otimes_1 b)c.$$

Besides the monomorphism

$$J^1 : \mathcal{A} \ni a \mapsto \mathbf{1} \otimes_1 a \in \mathcal{J}^1$$

(1.2.16), there is the $\mathcal{A}$-module monomorphism

$$i_1 : \mathcal{A} \ni a \mapsto a \otimes_1 \mathbf{1} \in \mathcal{J}^1.$$

With these monomorphisms, we have the canonical $\mathcal{A}$-module splitting

$$\begin{aligned} &\mathcal{J}^1 = i_1(\mathcal{A}) \oplus \mathcal{O}^1, \\ &aJ^1(b) = a \otimes_1 b = ab \otimes_1 \mathbf{1} + a(\mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1}), \end{aligned} \tag{1.3.2}$$

where the $\mathcal{A}$-module $\mathcal{O}^1$ is generated by the elements $\mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1}$ for all $b \in \mathcal{A}$. Let us consider the corresponding $\mathcal{A}$-module epimorphism

$$h^1 : \mathcal{J}^1 \ni \mathbf{1} \otimes_1 b \mapsto \mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1} \in \mathcal{O}^1 \tag{1.3.3}$$

and the composition

$$d^1 = h^1 \circ J_1 : \mathcal{A} \ni b \mapsto \mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1} \in \mathcal{O}^1, \tag{1.3.4}$$

which is a $\mathcal{K}$-module morphism. This is a $\mathcal{O}^1$-valued derivation of the $\mathcal{K}$-ring $\mathcal{A}$ which obeys the Leibniz rule

$$d^1(ab) = \mathbf{1} \otimes_1 ab - ab \otimes_1 \mathbf{1} + a \otimes_1 b - a \otimes_1 b = ad^1b + (d^1a)b.$$

It follows from the relation (1.3.1) that $ad^1b = (d^1b)a$ for all $a, b \in \mathcal{A}$. Thus, seen as an $\mathcal{A}$-module, $\mathcal{O}^1$ is generated by the elements d^1a for all $a \in \mathcal{A}$.

Let $\mathcal{O}^{1*} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}^1, \mathcal{A})$ be the dual of the $\mathcal{A}$-module $\mathcal{O}^1$. In view of the splittings (1.2.12) and (1.3.2), the isomorphism (1.2.19) reduces to the duality relation

$$\mathfrak{d}\mathcal{A} = \mathcal{O}^{1*}, \tag{1.3.5}$$

$$\mathfrak{d}\mathcal{A} \ni u \leftrightarrow \phi_u \in \mathcal{O}^{1*}, \qquad \phi_u(d^1a) = u(a), \qquad a \in \mathcal{A}. \tag{1.3.6}$$

In a more direct way (see Proposition 4.2.1 below), the isomorphism (1.3.5) is derived from the facts that $\mathcal{O}^1$ is generated by elements d^1a, $a \in \mathcal{A}$, and that $\phi(d^1a)$ is a derivation of $\mathcal{A}$ for any $\phi \in \mathcal{O}^{1*}$. However, the morphism

$$\mathcal{O}^1 \to \mathcal{O}^{1**} = \mathfrak{d}\mathcal{A}^*$$

need not be an isomorphism.

Let us define the modules $\mathcal{O}^k$, $k = 2, \ldots$, as the exterior products of the $\mathcal{A}$-module $\mathcal{O}^1$. There are the higher degree generalizations

$$\begin{aligned} &h^k : \mathcal{J}^1(\mathcal{O}^{k-1}) \to \mathcal{O}^k, \\ &d^k = h^k \circ J^1 : \mathcal{O}^{k-1} \to \mathcal{O}^k \end{aligned} \tag{1.3.7}$$

of the morphisms (1.3.3) and (1.3.4). The operators (1.3.7) are nilpotent, i.e., $d^k \circ d^{k-1} = 0$. They form the cochain complex

$$0 \to \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{d^1} \mathcal{O}^1 \xrightarrow{d^2} \cdots \mathcal{O}^k \xrightarrow{d^{k+1}} \cdots . \tag{1.3.8}$$

Let us return to the first order jet module $\mathcal{J}^1(P)$ of an $\mathcal{A}$-module P. It is isomorphic to the tensor product

$$\mathcal{J}^1(P) = \mathcal{J}^1 \otimes P, \qquad (a \otimes_1 bp) \leftrightarrow (a \otimes_1 b) \otimes p. \tag{1.3.9}$$

Then the isomorphism (1.3.2) leads to the splitting

$$\begin{aligned} &\mathcal{J}^1(P) = (\mathcal{A} \oplus \mathcal{O}^1) \otimes P = (\mathcal{A} \otimes P) \oplus (\mathcal{O}^1 \otimes P), \\ &a \otimes_1 bp \leftrightarrow (ab + ad^1(b)) \otimes p. \end{aligned} \tag{1.3.10}$$

Applying the epimorphism π^1_0 (1.2.18) to this splitting, one obtains the short exact sequence of $\mathcal{A}$- and $\mathcal{A}^\bullet$-modules

$$\begin{aligned} &0 \longrightarrow \mathcal{O}^1 \otimes P \to \mathcal{J}^1(P) \xrightarrow{\pi^1_0} P \longrightarrow 0, \\ &(a \otimes_1 b - ab \otimes_1 \mathbf{1}) \otimes p \to (c \otimes_1 \mathbf{1} + a \otimes_1 b - ab \otimes_1 \mathbf{1}) \otimes p \to cp. \end{aligned} \tag{1.3.11}$$

This exact sequence is canonically split by the $\mathcal{A}^\bullet$-module morphism

$$P \ni ap \mapsto \mathbf{1} \otimes ap = a \otimes p + d^1(a) \otimes p \in \mathcal{J}^1(P).$$

However, it need not be split by an $\mathcal{A}$-module morphism, unless P is a projective $\mathcal{A}$-module.

DEFINITION 1.3.1: A *connection* on an $\mathcal{A}$-module P is defined as an $\mathcal{A}$-module morphism

$$\Gamma : P \to \mathcal{J}^1(P), \qquad \Gamma(ap) = a\Gamma(p), \tag{1.3.12}$$

which splits the exact sequence (1.3.11) or, equivalently, the exact sequence

$$0 \to \mathcal{O}^1 \otimes P \to (\mathcal{A} \oplus \mathcal{O}^1) \otimes P \to P \to 0. \tag{1.3.13}$$

□

If a splitting Γ (1.3.12) exists, it reads

$$J^1 p = \Gamma(p) + \nabla(p), \tag{1.3.14}$$

where ∇ is the complementary morphism

$$\nabla : P \to \mathcal{O}^1 \otimes P, \qquad \nabla(p) = \mathbf{1} \otimes_1 p - \Gamma(p). \tag{1.3.15}$$

Though this complementary morphism in fact is a *covariant differential* on the module P, it is traditionally called a connection on a module. It satisfies the *Leibniz rule*

$$\nabla(ap) = d^1 a \otimes p + a\nabla(p), \tag{1.3.16}$$

i.e., ∇ is an $(\mathcal{O}^1 \otimes P)$-valued first order differential operator on P. Thus, we come to the equivalent definition of a connection [50].

DEFINITION 1.3.2: A *connection* on an $\mathcal{A}$-module P is a $\mathcal{K}$-module morphism ∇ (1.3.15) which obeys the Leibniz rule (1.3.16). Sometimes, it is called the *Koszul connection*. □

The morphism ∇ (1.3.15) can be extended naturally to the morphism

$$\nabla : \mathcal{O}^1 \otimes P \to \mathcal{O}^2 \otimes P.$$

Then we have the morphism

$$R = \nabla^2 : P \to \mathcal{O}^2 \otimes P, \tag{1.3.17}$$

called the *curvature* of the connection ∇ on a module P.

In view of the isomorphism (1.3.5), any connection in Definition 1.3.2 determines a connection in the following sense.

DEFINITION 1.3.3: A *connection* on an $\mathcal{A}$-module P is an $\mathcal{A}$-module morphism

$$\mathfrak{d}\mathcal{A} \ni u \mapsto \nabla_u \in \mathrm{Diff}_1(P, P) \tag{1.3.18}$$

such that the first order differential operators ∇_u obey the *Leibniz rule*

$$\nabla_u(ap) = u(a)p + a\nabla_u(p), \quad a \in \mathcal{A}, \quad p \in P. \tag{1.3.19}$$

□

Definitions 1.3.2 and 1.3.3 are equivalent if $\mathcal{O}^1 = \mathfrak{d}\mathcal{A}^*$.

The *curvature* of the connection (1.3.18) is defined as a zero order differential operator

$$R(u, u') = [\nabla_u, \nabla_{u'}] - \nabla_{[u,u']} \tag{1.3.20}$$

on the module P for all $u, u' \in \mathfrak{d}\mathcal{A}$.

Let P be a commutative $\mathcal{A}$-ring and $\mathfrak{d}P$ the derivation module of P as a $\mathcal{K}$-ring. Definition 1.3.3 is modified as follows.

DEFINITION 1.3.4: A *connection* on an $\mathcal{A}$-ring P is an $\mathcal{A}$-module morphism

$$\mathfrak{d}\mathcal{A} \ni u \mapsto \nabla_u \in \mathfrak{d}P, \tag{1.3.21}$$

which is a connection on P as an $\mathcal{A}$-module, i.e., obeys the Leinbniz rule (1.3.19). □

Two such connections ∇_u and ∇'_u differ from each other in a derivation of the $\mathcal{A}$-ring P, i.e., which vanishes on $\mathcal{A} \subset P$. The curvature of the connection (1.3.21) is given by the formula (1.3.20).

1.4 Differential calculus over a commutative ring

In a general setting, the de Rham complex is defined as a cochain complex which is also a differential graded algebra. By a gradation throughout this Section is meant the $\mathbb{N}$-gradation.

A *graded algebra* Ω^* over a commutative ring $\mathcal{K}$ is defined as a direct sum

$$\Omega^* = \bigoplus_k \Omega^k$$

of $\mathcal{K}$-modules Ω^k, provided with an associative multiplication law $\alpha\cdot\beta$, $\alpha,\beta\in\Omega^*$, such that $\alpha\cdot\beta\in\Omega^{|\alpha|+|\beta|}$, where $|\alpha|$ denotes the degree of an element $\alpha\in\Omega^{|\alpha|}$. In particular, it follows that Ω^0 is a (non-commutative) $\mathcal{K}$-algebra $\mathcal{A}$, while $\Omega^{k>0}$ are $\mathcal{A}$-bimodules and Ω^* is an $(\mathcal{A}-\mathcal{A})$-algebra. A graded algebra is said to be *graded commutative* if

$$\alpha\cdot\beta = (-1)^{|\alpha||\beta|}\beta\cdot\alpha, \qquad \alpha,\beta\in\Omega^*.$$

A graded algebra Ω^* is called a *differential graded algebra* if it is a cochain complex of $\mathcal{K}$-modules

$$0\to\mathcal{K}\longrightarrow\mathcal{A}\stackrel{\delta}{\longrightarrow}\Omega^1\stackrel{\delta}{\longrightarrow}\cdots\Omega^k\stackrel{\delta}{\longrightarrow}\cdots \tag{1.4.1}$$

with respect to a coboundary operator δ which obeys the *graded Leibniz rule*

$$\delta(\alpha\cdot\beta) = \delta\alpha\cdot\beta + (-1)^{|\alpha|}\alpha\cdot\delta\beta. \tag{1.4.2}$$

In particular, $\delta:\mathcal{A}\to\Omega^1$ is a Ω^1-valued derivation of a $\mathcal{K}$-algebra $\mathcal{A}$.

The cochain complex (1.4.1) is the above mentioned *de Rham complex* of the differential graded algebra (Ω^*,δ). This algebra is also said to be a *differential calculus* over $\mathcal{A}$. Cohomology $H^*(\Omega^*)$ of the complex (1.4.1) is called the *de Rham cohomology* of a differential graded algebra. It is a graded algebra with respect to the *cup-product*

$$[\alpha]\smile[\beta] = [\alpha\cdot\beta], \tag{1.4.3}$$

where $[\alpha]$ denotes the de Rham cohomology class of elements $\alpha\in\Omega^*$.

A morphism γ between two differential graded algebras (Ω^*,δ) and (Ω'^*,δ') is defined as a cochain morphism, i.e., $\gamma\circ\delta=\gamma\circ\delta'$. It yields the corresponding morphism of the de Rham cohomology groups of these algebras.

One considers the minimal differential graded subalgebra $\Omega^*\mathcal{A}$ of the differential graded algebra Ω^* which contains $\mathcal{A}$. Seen as an $(\mathcal{A}-\mathcal{A})$-algebra, it is generated by the elements δa, $a\in\mathcal{A}$, and consists of monomials

$$\alpha = a_0\delta a_1\cdots\delta a_k, \qquad a_i\in\mathcal{A},$$

whose product obeys the *juxtaposition rule*

$$(a_0\delta a_1)\cdot(b_0\delta b_1) = a_0\delta(a_1 b_0)\cdot\delta b_1 - a_0 a_1\delta b_0\cdot\delta b_1$$

in accordance with the equality (1.4.2). The differential graded algebra $(\Omega^*\mathcal{A}, \delta)$ is called the *minimal differential calculus* over $\mathcal{A}$.

Let us show that any commutative $\mathcal{K}$-ring $\mathcal{A}$ defines a differential calculus.

As was mentioned above, the derivation module $\mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ is also a Lie $\mathcal{K}$-algebra. Let us consider the extended Chevalley–Eilenberg complex

$$0 \to \mathcal{K} \xrightarrow{\text{in}} C^*[\mathfrak{d}\mathcal{A}; \mathcal{A}] \tag{1.4.4}$$

of the Lie algebra $\mathfrak{d}\mathcal{A}$ with coefficients in the ring $\mathcal{A}$, regarded as a $\mathfrak{d}\mathcal{A}$-module [41]. This complex contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of $\mathcal{A}$-multilinear skew-symmetric maps

$$\phi^k : \overset{k}{\times} \mathfrak{d}\mathcal{A} \to \mathcal{A} \tag{1.4.5}$$

with respect to the Chevalley–Eilenberg coboundary operator

$$\begin{aligned} d\phi(u_0, \ldots, u_k) = & \sum_{i=0}^{k} (-1)^i u_i(\phi(u_0, \ldots, \widehat{u_i}, \ldots, u_k)) + \\ & \sum_{i<j} (-1)^{i+j} \phi([u_i, u_j], u_0, \ldots, \widehat{u}_i, \ldots, \widehat{u}_j, \ldots, u_k). \end{aligned} \tag{1.4.6}$$

Indeed, a direct verification shows that if ϕ is an $\mathcal{A}$-multilinear map, so is $d\phi$. In particular,

$$(da)(u) = u(a), \qquad a \in \mathcal{A}, \qquad u \in \mathfrak{d}\mathcal{A}, \tag{1.4.7}$$

$$\begin{aligned} (d\phi)(u_0, u_1) = & u_0(\phi(u_1)) - u_1(\phi(u_0)) \\ & -\phi([u_0, u_1]), \qquad \phi \in \mathcal{O}^1[\mathfrak{d}\mathcal{A}], \end{aligned} \tag{1.4.8}$$

$$\mathcal{O}^0[\mathfrak{d}\mathcal{A}] = \mathcal{A}, \qquad \mathcal{O}^1[\mathfrak{d}\mathcal{A}] = \mathrm{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A}, \mathcal{A}).$$

It follows that $d(\mathbf{1}) = 0$ and d is a $\mathcal{O}^1[\mathfrak{d}\mathcal{A}]$-valued derivation of $\mathcal{A}$.

The graded module $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is provided with the structure of a graded $\mathcal{A}$-algebra with respect to the product

$$\begin{aligned} & \phi \wedge \phi'(u_1, ..., u_{r+s}) = \\ & \qquad \sum_{i_1<\cdots<i_r; j_1<\cdots<j_s} \mathrm{sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s} \phi(u_{i_1}, \ldots, u_{i_r}) \phi'(u_{j_1}, \ldots, u_{j_s}), \\ & \phi \in \mathcal{O}^r[\mathfrak{d}\mathcal{A}], \qquad \phi' \in \mathcal{O}^s[\mathfrak{d}\mathcal{A}], \qquad u_k \in \mathfrak{d}\mathcal{A}, \end{aligned} \tag{1.4.9}$$

where $\mathrm{sgn}^{\cdots}_{\cdots}$ is the sign of a permutation. This product obeys the relations

$$d(\phi \wedge \phi') = d(\phi) \wedge \phi' + (-1)^{|\phi|} \phi \wedge d(\phi'), \quad \phi, \phi' \in \mathcal{O}^*[\mathfrak{d}\mathcal{A}], \tag{1.4.10}$$

$$\phi \wedge \phi' = (-1)^{|\phi||\phi'|} \phi' \wedge \phi. \tag{1.4.11}$$

By virtue of the first one, $(\mathcal{O}^*[\mathfrak{d}\mathcal{A}], d)$ is a differential graded $\mathcal{K}$-algebra, called the *Chevalley–Eilenberg differential calculus* over a $\mathcal{K}$-ring $\mathcal{A}$ [41]. The relation (1.4.11) shows that $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is a graded commutative algebra.

The *minimal Chevalley–Eilenberg differential calculus* $\mathcal{O}^*\mathcal{A}$ over a ring $\mathcal{A}$ consists of the monomials

$$a_0 da_1 \wedge \cdots \wedge da_k, \qquad a_i \in \mathcal{A}.$$

Its complex

$$0 \to \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{d} \mathcal{O}^1\mathcal{A} \xrightarrow{d} \cdots \mathcal{O}^k\mathcal{A} \xrightarrow{d} \cdots \tag{1.4.12}$$

is exactly the cochain complex (1.3.8). Indeed, comparing the equalities (1.3.6) and (1.4.7) shows that $d^1 = d$ on the $\mathcal{A}$-module

$$\mathcal{O}^1\mathcal{A} = \mathcal{O}^1 \subseteq \mathcal{O}^1[\mathfrak{d}\mathcal{A}] = \mathfrak{d}\mathcal{A}^*,$$

generated by elements d^1a, $a \in \mathcal{A}$. The complex (1.4.12) is called the *de Rham complex* of the $\mathcal{K}$-ring $\mathcal{A}$, and its cohomology $H^*(\mathcal{A})$ is said to be the *de Rham cohomology* of $\mathcal{A}$. This cohomology is a graded commutative algebra with respect to the cup-product (1.4.3) induced by the exterior product $\wedge$ of elements of $\mathcal{O}^*\mathcal{A}$ so that

$$[\phi] \smile [\phi'] = [\phi \wedge \phi'], \qquad [\phi] \smile [\phi'] = (-1)^{|[\phi]||[\phi']|}[\phi'] \smile [\phi]. \tag{1.4.13}$$

1.5 Local-ringed spaces

Local-ringed spaces are sheafs of local rings. For instance, smooth manifolds, represented by sheaves of real smooth functions, make up a subcategory of the category of local-ringed spaces.

A sheaf $\mathfrak{R}$ on a topological space X is said to be a *ringed space* if its stalk $\mathfrak{R}_x$ at each point $x \in X$ is a commutative ring [84]. A ringed space is often denoted by a pair $(X, \mathfrak{R})$ of a topological space X and a sheaf $\mathfrak{R}$ of rings on X which are called the *body* and the *structure sheaf* of a ringed space, respectively.

A ringed space is said to be a *local-ringed space* (a *geometric space* in the terminology of [84]) if it is a sheaf of local rings.

For instance, the sheaf C^0_X of continuous real functions on a topological space X is a local-ringed space. Its stalk C^0_x, $x \in X$, contains the unique maximal ideal of germs of functions vanishing at x.

Morphisms of local-ringed spaces are defined to be particular morphisms of sheaves on different topological spaces as follows.

Let $\varphi : X \to X'$ be a continuous map. Given a sheaf S on X, its *direct image* $\varphi_* S$ on X' is generated by the presheaf of assignments

$$X' \supset U' \mapsto S(\varphi^{-1}(U'))$$

for any open subset $U' \subset X'$. Conversely, given a sheaf S' on X', its *inverse image* $\varphi^* S'$ on X is defined as the pull-back onto X of the topological fibre bundle S' over X', i.e., $\varphi^* S'_x = S_{\varphi(x)}$. This sheaf is generated by the presheaf which associates to any open $V \subset X$ the direct limit of modules $S'(U)$ over all open subsets $U \subset X'$ such that $V \subset f^{-1}(U)$.

Example 1.5.1: Let $i : X \to X'$ be a closed subspace of X'. Then $i_* S$ is a unique sheaf on X' such that

$$i_* S|_X = S, \qquad i_* S|_{X' \setminus X} = 0.$$

Indeed, if $x' \in X \subset X'$, then $i_* S(U') = S(U' \cap X)$ for any open neighborhood U of this point. If $x' \notin X$, there exists its neighborhood U' such that $U' \cap X$ is empty, i.e., $i_* S(U') = 0$. The sheaf $i_* S$ is called the *trivial extension* of the sheaf S. $\diamondsuit$

By a *morphism of ringed spaces* $(X, \mathfrak{R}) \to (X', \mathfrak{R}')$ is meant a pair (φ, Φ) of a continuous map $\varphi : X \to X'$ and a sheaf morphism $\Phi : \mathfrak{R}' \to \varphi_* \mathfrak{R}$ or, equivalently, a sheaf morphisms $\varphi^* \mathfrak{R}' \to \mathfrak{R}$ [84]. Restricted to each stalk, a sheaf morphism Φ is assumed to be a ring homomorphism. A morphism of ringed spaces is said to be:

- a monomorphism if φ is an injection and Φ is an epimorphism,
- an epimorphism if φ is a surjection, while Φ is a monomorphism.

Let $(X, \mathfrak{R})$ be a local-ringed space. By a *sheaf* $\mathfrak{d}\mathfrak{R}$ *of derivations* of the sheaf $\mathfrak{R}$ is meant a subsheaf of endomorphisms of $\mathfrak{R}$ such that any section u of $\mathfrak{d}\mathfrak{R}$ over an open subset $U \subset X$ is a derivation of the ring $\mathfrak{R}(U)$. It should be emphasized that, since (5.3.1) is not necessarily an isomorphism, a derivation of the ring $\mathfrak{R}(U)$ need not be a section of the sheaf $\mathfrak{d}\mathfrak{R}|_U$. Namely, it may happen that, given open sets $U' \subset U$, there is no restriction morphism

$$\mathfrak{d}(\mathfrak{R}(U)) \to \mathfrak{d}(\mathfrak{R}(U')).$$

Given a local-ringed space $(X, \mathfrak{R})$, a sheaf P on X is called a *sheaf of* $\mathfrak{R}$*-modules* if every stalk P_x, $x \in X$, is an $\mathfrak{R}_x$-module or, equivalently, if $P(U)$ is an $\mathfrak{R}(U)$-module for any open subset $U \subset X$. A sheaf of $\mathfrak{R}$-modules P is said to be *locally free* if there exists an open neighborhood U of every point $x \in X$ such that $P(U)$ is a free $\mathfrak{R}(U)$-module. If all these free modules are of finite rank (resp. of the same finite rank), one says that P is of *finite type* (resp. of constant rank). The structure module of a locally free sheaf is called a *locally free module.*

The following is a generalization of Proposition 5.3.8 [47].

PROPOSITION 1.5.1: Let X be a paracompact space which admits a partition of unity by elements of the structure module $S(X)$ of some sheaf S of real functions on X. Let P be a sheaf of S-modules. Then P is fine and, consequently, acyclic. $\Box$

1.6 Differential geometry of $C^\infty(X)$-modules

The sheaf C^∞_X of smooth real functions on a smooth manifold X provides an important example of a local-ringed spaces.

Remark 1.6.1: Throughout the Lectures, *smooth manifolds* are finite-dimensional real manifolds, though infinite-dimensional Banach and Hilbert manifolds are also smooth. A smooth real manifold is customarily assumed to be Hausdorff and *second-countable* (i.e., it has a countable base for topology). Consequently, it is a locally compact space which is a union of a countable number of compact subsets, a *separable* space (i.e., it has a countable dense subset), a paracompact and completely regular space. Being paracompact, a smooth manifold admits a partition of unity by smooth real functions. One also can show that, given two disjoint closed subsets N and N' of a smooth manifold X, there exists a smooth function f on X such that $f|_N = 0$ and $f|_{N'} = 1$. Unless otherwise stated, manifolds are assumed to be connected and, consequently, arcwise connected. We follow the notion of a manifold without boundary. $\diamondsuit$

Similarly to the sheaf C^0_X of continuous functions, the stalk C^∞_x of the sheaf C^∞_X at a point $x \in X$ has a unique maximal ideal of germs of smooth functions vanishing at x. Though the sheaf C^∞_X is defined on a topological space X, it fixes a unique smooth manifold structure on X as follows.

THEOREM 1.6.1: Let X be a paracompact topological space and $(X, \mathfrak{R})$ a local-ringed space. Let X admit an open cover $\{U_i\}$ such that the sheaf $\mathfrak{R}$ restricted to each U_i is isomorphic to the local-ringed space $(\mathbb{R}^n, C^\infty_{R^n})$. Then X is an n-dimensional smooth manifold together with a natural isomorphism of local-ringed spaces $(X, \mathfrak{R})$ and (X, C^∞_X). $\Box$

One can think of this result as being an alternative definition of smooth real manifolds in terms of local-ringed spaces. In particular, there is one-to-one correspondence between smooth manifold morphisms $X \to X'$ and the $\mathbb{R}$-ring morphisms $C^\infty(X') \to C^\infty(X)$.

Remark 1.6.2: Let $X \times X'$ be a manifold product. The ring $C^\infty(X \times X')$ is constructed from the rings $C^\infty(X)$ and $C^\infty(X')$ as follows. Whenever referring to a topology on the ring $C^\infty(X)$, we will mean the topology of compact convergence for all derivatives [70]. The $C^\infty(X)$ is a *Fréchet ring* with respect to this topology, i.e., a complete metrizable locally convex topological vector space. There is an isomorphism of Fréchet rings

$$C^\infty(X)\widehat{\otimes}C^\infty(X') \cong C^\infty(X \times X'), \tag{1.6.1}$$

where the left-hand side, called the *topological tensor product*, is the completion of $C^\infty(X) \otimes C^\infty(X')$ with respect to Grothendieck's topology, defined as follows. If E_1 and E_2 are locally convex topological vector spaces, *Grothendieck's topology* is the finest locally convex topology on $E_1 \otimes E_2$ such that the canonical mapping of $E_1 \times E_2$ to $E_1 \otimes E_2$ is continuous [70]. It is also called the π-topology in contrast with the coarser ε-topology on $E_1 \otimes E_2$ [65, 85]. Furthermore, for any two open subsets $U \subset X$ and $U' \subset X'$, let us consider the topological tensor product of rings

$C^\infty(U)\widehat{\otimes}C^\infty(U')$. These tensor products define a locally ringed space $(X\times X', C^\infty_X\widehat{\otimes}C^\infty_{X'})$. Due to the isomorphism (1.6.1) written for all $U\subset X$ and $U'\subset X'$, we obtain the sheaf isomorphism

$$C^\infty_X\widehat{\otimes}C^\infty_{X'} = C^\infty_{X\times X'}. \tag{1.6.2}$$

$\diamondsuit$

Since a smooth manifold admits a partition of unity by smooth functions, it follows from Proposition 1.5.1 that any sheaf of C^∞_X-modules on X is fine and, consequently, acyclic.

For instance, let $Y\to X$ be a smooth (finite-dimensional) vector bundle. The germs of its sections make up a sheaf of C^∞_X-modules, called the *structure sheaf* S_Y of a vector bundle $Y\to X$. The sheaf S_Y is fine.

In particular, all sheafs $\mathcal{O}^k_X$, $k\in\mathbb{N}_+$, of germs of exterior forms on X is fine. These sheaves constitute the *de Rham complex*

$$0\to\mathbb{R}\longrightarrow C^\infty_X\xrightarrow{d}\mathcal{O}^1_X\xrightarrow{d}\cdots\mathcal{O}^k_X\xrightarrow{d}\cdots. \tag{1.6.3}$$

The corresponding complex of structure modules of these sheaves is the *de Rham complex*

$$0\to\mathbb{R}\longrightarrow C^\infty(X)\xrightarrow{d}\mathcal{O}^1(X)\xrightarrow{d}\cdots\mathcal{O}^k(X)\xrightarrow{d}\cdots \tag{1.6.4}$$

of exterior forms on a manifold X. Its cohomology is called the *de Rham cohomology* $H^*(X)$ of X. Due to the Poincaré lemma, the complex (1.6.3) is exact and, thereby, is a fine resolution of the constant sheaf $\mathbb{R}$ on a manifold. Then a corollary of Theorem 5.3.6 is the classical *de Rham theorem*.

THEOREM 1.6.2: There is the isomorphism

$$H^k(X) = H^k(X;\mathbb{R}) \tag{1.6.5}$$

of the de Rham cohomology $H^*(X)$ of a manifold X to cohomology of X with coefficients in the constant sheaf $\mathbb{R}$. $\Box$

Remark 1.6.3: Let us consider the short exact sequence of constant sheaves

$$0\to\mathbb{Z}\longrightarrow\mathbb{R}\longrightarrow U(1)\to 0, \tag{1.6.6}$$

where $U(1)=\mathbb{R}/\mathbb{Z}$ is the circle group of complex numbers of unit modulus. This exact sequence yields the long exact sequence of the sheaf cohomology groups

$$\begin{aligned}0\to\mathbb{Z}\longrightarrow\mathbb{R}\longrightarrow U(1)\longrightarrow H^1(X;\mathbb{Z})\longrightarrow H^1(X;\mathbb{R})\longrightarrow\cdots\\ H^p(X;\mathbb{Z})\longrightarrow H^p(X;\mathbb{R})\longrightarrow H^p(X;U(1))\longrightarrow H^{p+1}(X;\mathbb{Z})\longrightarrow\cdots,\end{aligned}$$

where

$$H^0(X;\mathbb{Z})=\mathbb{Z},\qquad H^0(X;\mathbb{R})=\mathbb{R}$$

and $H^0(X;U(1)) = U(1)$. This exact sequence defines the homomorphism

$$H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{R}) \tag{1.6.7}$$

of cohomology with coefficients in the constant sheaf $\mathbb{Z}$ to that with coefficients in $\mathbb{R}$. Combining the isomorphism (1.6.5) and the homomorphism (1.6.7) leads to the cohomology homomorphism

$$H^*(X;\mathbb{Z}) \to H^*(X). \tag{1.6.8}$$

Its kernel contains all cyclic elements of cohomology groups $H^k(X;\mathbb{Z})$. $\diamondsuit$

Given a vector bundle $Y \to X$, the structure module of the sheaf S_Y coincides with the *structure module* $Y(X)$ of global sections of $Y \to X$. The *Serre–Swan theorem*, shows that these modules exhaust all projective modules of finite rank over $C^\infty(X)$. This theorem originally has been proved in the case of a compact manifold X, but it is generalized to an arbitrary smooth manifold [41].

THEOREM 1.6.3: Let X be a smooth manifold. A $C^\infty(X)$-module P is isomorphic to the structure module of a smooth vector bundle over X iff it is a projective module of finite rank. $\Box$

• The structure module $Y^*(X)$ of the dual $Y^* \to X$ of a vector bundle $Y \to X$ is the $C^\infty(X)$-dual $Y(X)^*$ of the structure module $Y(X)$ of $Y \to X$.

• Any exact sequence of vector bundles

$$0 \to Y \longrightarrow Y' \longrightarrow Y'' \to 0 \tag{1.6.9}$$

over the same base X yields the exact sequence

$$0 \to Y(X) \longrightarrow Y'(X) \longrightarrow Y''(X) \to 0 \tag{1.6.10}$$

of their structure modules, and *vice versa*. In accordance with the well-known theorem [61, 79], the exact sequence (1.6.9) is always split. Every its splitting defines that of the exact sequence (1.6.10), and *vice versa*.

• For instance, the derivation module of the $\mathbb{R}$-ring $C^\infty(X)$ coincides with the $C^\infty(X)$-module $\mathcal{T}_1(X)$ of vector fields on X, i.e., with the structure module of the tangent bundle TX of X. Hence, it is a projective $C^\infty(X)$-module of finite rank. It is the $C^\infty(X)$-dual $\mathcal{T}_1(X) = \mathcal{O}^1(X)^*$ of the structure module $\mathcal{O}^1(X)$ of the cotangent bundle T^*X of X which is the module of differential one-forms on X and, conversely, $\mathcal{O}^1(X) = \mathcal{T}_1(X)^*$. It follows that the Chevalley–Eilenberg differential calculus over the $\mathbb{R}$-ring $C^\infty(X)$ is exactly the differential graded algebra $(\mathcal{O}^*(X), d)$ of exterior forms on X, where the Chevalley–Eilenberg coboundary operator d (1.4.6) coincides with the exterior differential. Accordingly, the de Rham complex (1.4.12) of the $\mathbb{R}$-ring $C^\infty(X)$ is the de Rham complex (1.6.4) of exterior forms on X. Moreover, one can show that $(\mathcal{O}^*(X), d)$ is a minimal

differential calculus, i.e., the $C^\infty(X)$-module $\mathcal{O}^1(X)$ is generated by elements df, $f\in C^\infty(X)$. Indeed, using the notation in the proof of Theorem 1.6.3, one can write

$$\mathcal{O}^1(X)\ni\phi=\sum_\xi l_\xi^2\phi=\sum_\xi l_\xi^2\phi_\mu dx^\mu=\sum_\xi(l_\xi\phi_\mu d(l_\xi x^\mu)-l_\xi\phi_\mu x^\mu dl_\xi), \tag{1.6.11}$$

where (x^μ) are local coordinates on U_ξ and $l_\xi x^\mu$ and l_ξ are functions on X.

Remark 1.6.4: Let us note that the above mentioned Chevalley–Eilenberg differential calculus over the $\mathbb{R}$-ring $C^\infty(X)$ is a subcomplex of the Chevalley–Eilenberg complex of the Lie algebra $\mathcal{T}_1(X)$ with coefficients in $C^\infty(X)$. It consists of skew-symmetric morphisms of $\mathcal{T}_1(X)$ to $C^\infty(X)$ which are not only $\mathbb{R}$-multilinear, but $C^\infty(X)$-multilinear. The Chevalley–Eilenberg cohomology of smooth vector fields with coefficients in a trivial representation and in spaces of smooth tensor fields has been studied in detail [39] $\diamondsuit$

• Let $Y\to X$ be a vector bundle and $Y(X)$ its structure module. The r-order jet manifold J^rY of $Y\to X$ consists of the equivalence classes j^r_xs, $x\in X$, of sections s of $Y\to X$ which are identified by the $r+1$ terms of their Taylor series at points $x\in X$. Since $Y\to X$ is a vector bundle, so is the jet bundle $J^rY\to X$. Its structure module $J^rY(X)$ is exactly the r-order jet module $\mathcal{J}^r(Y(X))$ of the $C^\infty(X)$-module $Y(X)$ in Section 1.2 [51]. As a consequence, the notion of a connection on the structure module $Y(X)$ is equivalent to the standard geometric notion of a connection on a vector bundle $Y\to X$ [61]. Indeed, connection on a fibre bundle $Y\to X$ is defined as a global section Γ of the affine jet bundle $J^1Y\to Y$. If $Y\to X$ is a vector bundle, there exists the exact sequence

$$0\to T^*X\underset{X}{\otimes}Y\longrightarrow J^1Y\longrightarrow Y\to 0 \tag{1.6.12}$$

over X which is split by Γ. Conversely, any slitting of this exact sequence yields a connection $Y\to X$. The exact sequence of vector bundles (1.6.12) induces the exact sequence of their structure modules

$$0\to\mathcal{O}^1(X)\otimes Y(X)\longrightarrow J^1Y(X)\longrightarrow Y(X)\to 0. \tag{1.6.13}$$

Then any connection Γ on a vector bundle $Y\to X$ defines a splitting of the exact sequence (1.6.13) which, by Definition 1.3.1, is a connection on the $C^\infty(X)$-module $Y(X)$, and *vice versa*.

Let now P be an arbitrary $C^\infty(X)$-module. One can reformulate Definitions 1.3.2 and 1.3.3 of a connection on P as follows.

DEFINITION 1.6.4: A connection on a $C^\infty(X)$-module P is a $C^\infty(X)$-module morphism

$$\nabla: P\to\mathcal{O}^1(X)\otimes P, \tag{1.6.14}$$

which satisfies the Leibniz rule

$$\nabla(fp)=df\otimes p+f\nabla(p),\qquad f\in C^\infty(X),\qquad p\in P.$$

□

DEFINITION 1.6.5: A connection on a $C^\infty(X)$-module P associates to any vector field $\tau \in \mathcal{T}_1(X)$ on X a first order differential operator ∇_τ on P which obeys the Leibniz rule

$$\nabla_\tau(fp) = (\tau\rfloor df)p + f\nabla_\tau p. \tag{1.6.15}$$

□

Since $\mathcal{O}^1(X) = \mathcal{T}_1(X)^*$, Definitions 1.6.4 and 1.6.5 are equivalent.

Let us note that a connection on an arbitrary $C^\infty(X)$-module need not exist, unless it is a projective or locally free module (see Theorem 1.7.4 below).

The curvature of a connection ∇ in Definitions 1.6.4 and 1.6.5 is defined as the zero-order differential operator

$$R(\tau,\tau') = [\nabla_\tau, \nabla_{\tau'}] - \nabla_{[\tau,\tau']} \tag{1.6.16}$$

on a module P for all vector fields $\tau, \tau' \in \mathcal{T}_1(X)$ on X.

1.7 Connections on local-ringed spaces

Let $(X,\mathfrak{R})$ be a local-ringed space and $\mathfrak{P}$ a sheaf of $\mathfrak{R}$-modules on X. For any open subset $U \subset X$, let us consider the jet module $\mathcal{J}^1(\mathfrak{P}(U))$ of the module $\mathfrak{P}(U)$. It consists of the elements of $\mathfrak{R}(U) \otimes \mathfrak{P}(U)$ modulo the pointwise relations (1.2.14). Hence, there is the restriction morphism

$$\mathcal{J}^1(\mathfrak{P}(U)) \to \mathcal{J}^1(\mathfrak{P}(V))$$

for any open subsets $V \subset U$, and the jet modules $\mathcal{J}^1(\mathfrak{P}(U))$ constitute a presheaf. This presheaf defines the *sheaf* $\mathfrak{J}^1\mathfrak{P}$ *of jets* of $\mathfrak{P}$ (or simply the *jet sheaf*). The jet sheaf $\mathfrak{J}^1\mathfrak{R}$ of the sheaf $\mathfrak{R}$ of local rings is introduced in a similar way. Since the relations (1.2.14) and (1.3.1) on the ring $\mathfrak{R}(U)$ and modules $\mathfrak{P}(U)$, $\mathcal{J}^1(\mathfrak{P}(U))$, $\mathcal{J}^1(\mathfrak{R}(U))$ are pointwise relations for any open subset $U \subset X$, they commute with the restriction morphisms. Therefore, the direct limits of the quotients modulo these relations exist [62]. Then we have the sheaf $\mathcal{O}^1\mathfrak{R}$ of one-forms over the sheaf $\mathfrak{R}$, the sheaf isomorphism

$$\mathfrak{J}^1(\mathfrak{P}) = (\mathfrak{R} \oplus \mathcal{O}^1\mathfrak{R}) \otimes \mathfrak{P},$$

and the exact sequences of sheaves

$$0 \to \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P} \to \mathfrak{J}^1(\mathfrak{P}) \to \mathfrak{P} \to 0, \tag{1.7.1}$$

$$0 \to \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P} \to (\mathfrak{R} \oplus \mathcal{O}^1\mathfrak{R}) \otimes \mathfrak{P} \to \mathfrak{P} \to 0. \tag{1.7.2}$$

They reflect the quotient (1.3.3), the isomorphism (1.3.10) and the exact sequences of modules (1.3.11), (1.3.13), respectively.

Remark 1.7.1: It should be emphasized that, because of the inequality (5.3.1), the duality relation (1.3.5) is not extended to the sheaves $\mathfrak{d}\mathfrak{R}$ and $\mathcal{O}^1\mathfrak{R}$ in general, unless $\mathfrak{d}\mathfrak{R}$ and $\mathcal{O}^1$ are locally free sheaves of finite rank. If $\mathfrak{P}$ is a locally free sheaf of finite rank, so is $\mathfrak{J}^1\mathfrak{P}$. $\diamondsuit$

Following Definitions 1.3.1, 1.3.2 of a connection on modules, we come to the following notion of a connection on sheaves.

DEFINITION 1.7.1: Given a local-ringed space $(X, \mathfrak{R})$ and a sheaf $\mathfrak{P}$ of $\mathfrak{R}$-modules on X, a *connection* on a sheaf $\mathfrak{P}$ is defined as a splitting of the exact sequence (1.7.1) or, equivalently, the exact sequence (1.7.2). $\Box$

Theorem 5.3.3 leads to the following compatibility of the notion of a connection on sheaves with that of a connection on modules.

PROPOSITION 1.7.2: If there exists a connection on a sheaf $\mathfrak{P}$ in Definition 1.7.1, then there exists a connection on a module $\mathfrak{P}(U)$ for any open subset $U \subset X$. Conversely, if for any open subsets $V \subset U \subset X$ there are connections on the modules $\mathfrak{P}(U)$ and $\mathfrak{P}(V)$ related by the restriction morphism, then the sheaf $\mathfrak{P}$ admits a connection. $\Box$

Example 1.7.2: Let $Y \to X$ be a vector bundle. Every linear connection Γ on $Y \to X$ defines a connection on the structure module $Y(X)$ such that the restriction $\Gamma|_U$ is a connection on the module $Y(U)$ for any open subset $U \subset X$. Then we have a connection on the structure sheaf Y_X. Conversely, a connection on the structure sheaf Y_X defines a connection on the module $Y(X)$ and, consequently, a connection on the vector bundle $Y \to X$. $\diamondsuit$

As an immediate consequence of Proposition 1.7.2, we find that the exact sequence of sheaves (1.7.2) is split iff there exists a sheaf morphism

$$\nabla : \mathfrak{P} \to \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P}, \tag{1.7.3}$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s), \qquad f \in \mathcal{A}(U), \qquad s \in \mathfrak{P}(U),$$

for any open subset $U \in X$. It leads to the following equivalent definition of a connection on sheaves in the spirit of Definition 1.3.2.

DEFINITION 1.7.3: The sheaf morphism (1.7.3) is a *connection* on the sheaf $\mathfrak{P}$. $\Box$

Similarly to the case of connections on modules, the *curvature* of the connection (1.7.3) on a sheaf $\mathfrak{P}$ is given by the expression

$$R = \nabla^2 : \mathfrak{P} \to \mathcal{O}^2_X \otimes \mathfrak{P}. \tag{1.7.4}$$

The exact sequence (1.7.2) need not be split. One can obtain the following criteria of the existence of a connection on a sheaf.

Let $\mathfrak{P}$ be a locally free sheaf of $\mathfrak{R}$-modules. Then we have the exact sequence of sheaves

$$0 \to \mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P}) \to \mathrm{Hom}\,(\mathfrak{P}, (\mathfrak{R} \oplus \mathcal{O}^1\mathfrak{R}) \otimes \mathfrak{P}) \to \mathrm{Hom}\,(\mathfrak{P}, \mathfrak{P}) \to 0$$

and the corresponding exact sequence (5.3.16) of the cohomology groups

$$\begin{aligned} 0 \to H^0(X; \mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P})) \to H^0(X; \mathrm{Hom}\,(\mathfrak{P}, (\mathfrak{R} \oplus \mathcal{O}^1\mathfrak{R}) \otimes \mathfrak{P})) \to \\ H^0(X; \mathrm{Hom}\,(\mathfrak{P}, \mathfrak{P})) \to H^1(X; \mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P})) \to \cdots. \end{aligned}$$

The identity morphism $\mathrm{Id} : \mathfrak{P} \to \mathfrak{P}$ belongs to $H^0(X; \mathrm{Hom}\,(\mathfrak{P}, \mathfrak{P}))$. Its image in

$$H^1(X; \mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1\mathfrak{R} \otimes \mathfrak{P}))$$

is called the *Atiyah class*. If this class vanishes, there exists an element of

$$\mathrm{Hom}\,(\mathfrak{P}, (\mathfrak{R} \oplus \mathcal{O}^1\mathfrak{R}) \otimes \mathfrak{P}))$$

whose image is $\mathrm{Id}\,\mathfrak{P}$, i.e., a splitting of the exact sequence (1.7.2).

In particular, let X be a manifold and $\mathfrak{R} = C^\infty_X$ the sheaf of smooth functions on X. The sheaf $\mathfrak{d}C^\infty_X$ of its derivations is isomorphic to the sheaf of vector fields on a manifold X. It follows that:

- there is the restriction morphism $\mathfrak{d}(C^\infty(U)) \to \mathfrak{d}(C^\infty(V))$ for any open sets $V \subset U$,
- $\mathfrak{d}C^\infty_X$ is a locally free sheaf of C^∞_X-modules of finite rank,
- the sheaves $\mathfrak{d}C^\infty_X$ and $\mathcal{O}^1_X$ are mutually dual.

Let $\mathfrak{P}$ be a locally free sheaf of C^∞_X-modules. In this case, $\mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1_X \otimes \mathfrak{P})$ is a locally free sheaf of C^∞_X-modules. It is fine and acyclic. Its cohomology group

$$H^1(X; \mathrm{Hom}\,(\mathfrak{P}, \mathcal{O}^1_X \otimes \mathfrak{P}))$$

vanishes, and the exact sequence

$$0 \to \mathcal{O}^1_X \otimes \mathfrak{P} \to (C^\infty_X \oplus \mathcal{O}^1_X) \otimes \mathfrak{P} \to \mathfrak{P} \to 0 \tag{1.7.5}$$

admits a splitting. This proves the following.

PROPOSITION 1.7.4: Any locally free sheaf of C^∞_X-modules on a manifold X admits a connection and, in accordance with Proposition 1.7.2, any locally free $C^\infty(X)$-module does well. □

In conclusion, let us consider a sheaf S of commutative C^∞_X-rings on a manifold X. Basing on Definition 1.3.4, we come to the following notion of a connection on a sheaf S of commutative C^∞_X-rings.

DEFINITION 1.7.5: Any morphism

$$\mathfrak{d}C^\infty_X \ni \tau \mapsto \nabla_\tau \in \mathfrak{d}S,$$

which is a connection on S as a sheaf of C^∞_X-modules, is called a connection on the sheaf S of rings. $\Box$

Its curvature is given by the expression

$$R(\tau,\tau') = [\nabla_\tau, \nabla_{\tau'}] - \nabla_{[\tau,\tau']}, \tag{1.7.6}$$

similar to the expression (1.3.20) for the curvature of a connection on modules.

Chapter 2

Geometry of quantum systems

Algebraic quantum theory usually deals with Hilbert spaces. This Chapter addresses differential geometry of Banach and Hilbert manifolds and, in particular, Hilbert bundles and bundles of C^*-algebras over a smooth manifolds X. For instance, this is the case of time-dependent quantum systems (where $X = \mathbb{R}$) (Section 2.5) and quantum models depending on classical parameters (Section 2.6). Their differential geometry is similar to differential geometry of finite-dimensional smooth manifolds and bundles in main, and it is formulated in algebraic terms of differential geometry of modules and, in particular, $C^\infty(X)$-modules.

2.1 Geometry of Banach manifolds

We start with the notion of a real Banach manifold [55, 86]. Banach manifolds are defined similarly to finite-dimensional smooth manifolds, but they are modelled on Banach spaces, not necessarily finite-dimensional.

Let us recall some particular properties of (infinite-dimensional) real Banach spaces. Let us note that a finite-dimensional Banach space is always provided with an Euclidean norm.

• Given Banach spaces E and H, every continuous bijective linear map of E to H is an isomorphism of topological vector spaces.

• Given a Banach space E, let F be its closed subspace. One says that F *splits* in E if there exists a closed complement F' of F such that $E \cong F \oplus F'$. In particular, finite-dimensional and finite-codimensional subspaces split in E. As a consequence, any subspace of a finite-dimensional space splits.

• Let E and H be Banach spaces and $f : E \to H$ a continuous injection. One says that f *splits* if there exists an isomorphism

$$g : H \to H_1 \times H_2$$

such that $g \circ f$ yields an isomorphism of E onto $H_1 \times \{0\}$.

• Given Banach spaces $(E, \|.\|_E)$ and $(H, \|.\|_H)$, one can provide the set $\mathrm{Hom}^0(E, H)$ of continuous linear morphisms of E to H with the norm

$$||f|| = \sup_{||z||_E=1} ||f(z)||_H, \qquad f \in \mathrm{Hom}^0(E, H). \tag{2.1.1}$$

If E, H and F are Banach spaces, the bilinear map

$$\mathrm{Hom}^0(E, F) \times \mathrm{Hom}^0(F, H) \to \mathrm{Hom}^0(E, H),$$

obtained by the composition $f \circ g$ of morphisms $\gamma \in \mathrm{Hom}^0(E, F)$ and $f \in \mathrm{Hom}^0(F, H)$, is continuous. Let us note that this assertion is false for more general spaces, e.g., the Fréchet ones.

• Let $(E, \|.\|_E)$ and $(H, \|.\|_H)$ be real Banach spaces. One says that a continuous map $f : E \to H$ (not necessarily linear and isometric) is a *differentiable function* between E and H if, given a point $z \in E$, there exists an $\mathbb{R}$-linear continuous map

$$df(z) : E \to H$$

(not necessarily isometric) such that

$$f(z') = f(z) + df(z)(z' - z) + o(z' - z),$$
$$\lim_{\|z'-z\|_E \to 0} \frac{\|o(z' - z)\|_H}{\|z' - z\|_E} = 0,$$

for any z' in some open neighborhood U of z. For instance, any continuous linear morphism f of E to H is differentiable and $df(z)z = f(z)$. The linear map $df(z)$ is called a *differential* of f at a point $z \in U$. Given an element $v \in E$, we obtain the map

$$E \ni z \mapsto \partial_v f(z) = df(z)v \in H, \tag{2.1.2}$$

called the *derivative* of a function f along a vector $v \in E$. One says that f is two-times differentiable if the map (2.1.2) is differentiable for any $v \in E$. Similarly, r-times differentiable and infinitely differentiable (smooth) functions on a Banach space are defined. The composition of smooth maps is a smooth map.

The following *inverse mapping theorem* enables one to consider smooth Banach manifolds and bundles similarly to the finite-dimensional ones.

THEOREM 2.1.1: Let $f : E \to H$ be a smooth map such that, given a point $z \in E$, the differential $df(z) : E \to H$ is an isomorphism of topological vector spaces. Then f is a local isomorphism at z. □

Let us turn to the notion of a Banach manifold, without repeating the statements true both for finite-dimensional and Banach manifolds.

DEFINITION 2.1.2: A *Banach manifold* $\mathcal{B}$ modelled on a Banach space B is defined as a topological space which admits an atlas of charts $\Psi_{\mathcal{B}} = \{(U_\iota, \phi_\iota)\}$, where the maps ϕ_ι

are homeomorphisms of U_ι onto open subsets of the Banach space B, while the transition functions $\phi_\zeta\phi_\iota^{-1}$ from $\phi_\iota(U_\iota \cap U_\zeta) \subset B$ to $\phi_\zeta(U_\iota \cap U_\zeta) \subset B$ are smooth. Two atlases of a Banach manifold are said to be equivalent if their union is also an atlas. $\Box$

Unless otherwise stated, Banach manifolds are assumed to be connected paracompact Hausdorff topological spaces. A locally compact Banach manifold is necessarily finite-dimensional.

Remark 2.1.1: Let us note that a paracompact Banach manifold admits a smooth partition of unity iff its model Banach space does. For instance, this is the case of (real) separable Hilbert spaces. Therefore, we restrict our consideration to Hilbert manifolds modelled on separable Hilbert spaces. $\diamondsuit$

Any open subset U of a Banach manifold $\mathcal{B}$ is a Banach manifold whose atlas is the restriction of an atlas of $\mathcal{B}$ to U.

Morphisms of Banach manifolds are defined similarly to those of smooth finite-dimensional manifolds. However, the notion of the immersion and submersion need a certain modification (see Definition 2.1.3 below).

Tangent vectors to a smooth Banach manifold $\mathcal{B}$ are introduced by analogy with tangent vectors to a finite-dimensional one. Given a point $z \in \mathcal{B}$, let us consider the pair $(v; (U_\iota, \phi_\iota))$ of a vector $v \in B$ and a chart $(U_\iota \ni z, \phi_\iota)$ on a Banach manifold $\mathcal{B}$. Two pairs $(v; (U_\iota, \phi_\iota))$ and $(v'; (U_\zeta, \phi_\zeta))$ are said to be equivalent if

$$v' = d(\phi_\zeta\phi_\iota^{-1})(\phi_\iota(z))v. \qquad (2.1.3)$$

The equivalence classes of such pairs make up the *tangent space* $T_z\mathcal{B}$ to a Banach manifold $\mathcal{B}$ at a point $z \in \mathcal{B}$. This tangent space is isomorphic to the topological vector space B. Tangent spaces to a Banach manifold $\mathcal{B}$ are assembled into the *tangent bundle* $T\mathcal{B}$ of $\mathcal{B}$. It is a Banach manifold modelled over the Banach space $B \oplus B$ which possesses the transition functions

$$(\phi_\zeta\phi_\iota^{-1}, d(\phi_\zeta\phi_\iota^{-1})).$$

Any morphism $f : \mathcal{B} \to \mathcal{B}'$ of Banach manifolds yields the corresponding tangent morphism of the tangent bundles $Tf : T\mathcal{B} \to T\mathcal{B}'$.

Definition 2.1.3: Let $f : \mathcal{B} \to \mathcal{B}'$ be a morphism of Banach manifolds.

(i) It is called an immersion at a point $z \in \mathcal{B}$ if the tangent morphism Tf at z is injective and splits.

(ii) A morphism f is called a submersion at a point $z \in \mathcal{B}$ if Tf at z is surjective and its kernel splits. $\Box$

In the case of finite-dimensional smooth manifolds, the split conditions are superfluous, and Definition 2.1.3 recovers the notion of the immersion and submersion of smooth manifolds.

The range of a surjective submersion f of a Banach manifold is a submanifold, though f need not be an isomorphism onto a submanifold, unless f is an imbedding.

One can think of a surjective submersion $\pi : \mathcal{B} \to \mathcal{B}'$ of Banach manifolds as a *fibred Banach manifold*. For instance, the product $\mathcal{B}\times\mathcal{B}'$ of Banach manifolds is a fibred Banach manifold with respect to pr_1 and pr_2.

Let $\mathcal{B}$ be a Banach manifold and E a Banach space. The definition of a (locally trivial) vector bundle with the typical fibre E and the base $\mathcal{B}$ is a repetition of that of finite-dimensional smooth vector bundles. Such a vector bundle Y is a Banach manifold and $Y \to \mathcal{B}$ is a surjective submersion. The above mentioned tangent bundle $T\mathcal{B}$ of a Banach manifold exemplifies a vector bundle over $\mathcal{B}$.

Let **Bnh** be the *category of Banach spaces* and $\mathbf{Vect}(\mathcal{B})$ denotes the *category of vector bundles over a Banach manifold* $\mathcal{B}$ with respect to their morphisms over $\mathrm{Id}\,\mathcal{B}$. Let

$$F : \mathbf{Bnh} \times \mathbf{Bnh} \to \mathbf{Bnh} \tag{2.1.4}$$

be a functor of two variables which is covariant in the first and contravariant in the second. Then there exists a functor

$$VF : \mathbf{Vect}(\mathcal{B}) \times \mathbf{Vect}(\mathcal{B}) \to \mathbf{Vect}(\mathcal{B})$$

such that, if $Y_E \to \mathcal{B}$ and $Y_H \to \mathcal{B}$ are vector bundles with the typical fibres E and H, then $VF(Y_E, Y_H)$ is a vector bundle with the typical fibre $F(E,H)$. For instance, the Whitney sum, the tensor product, and the exterior product of vector bundles over a Banach manifold are defined in this way. In particular, since the topological dual E' of a Banach space E is a Banach space, one can associate to each vector bundle $Y_E \to \mathcal{B}$ the dual $Y_E^* = Y_{E'}$ with the typical fibre E'. For instance, the dual of the tangent bundle $T\mathcal{B}$ of a Banach manifold $\mathcal{B}$ is the *cotangent bundle* $T^*\mathcal{B}$.

Sections of the tangent bundle $T\mathcal{B} \to \mathcal{B}$ of a Banach manifold are called *vector fields* on a Banach manifold $\mathcal{B}$. They form a locally free module $\mathcal{T}_1(\mathcal{B})$ over the ring $C^\infty(\mathcal{B})$ of smooth real functions on $\mathcal{B}$. Every vector field ϑ on a Banach manifold $\mathcal{B}$ determines a derivation of the $\mathbb{R}$-ring $C^\infty(\mathcal{B})$ by the formula

$$f(z) \mapsto \partial_\vartheta f(z) = df(z)\vartheta(z), \qquad z \in \mathcal{B}.$$

Different vector fields yield different derivations. It follows that $\mathcal{T}_1(\mathcal{B})$ possesses a structure of a real Lie algebra, and there is its monomorphism

$$\mathcal{T}_1(\mathcal{B}) \to \mathfrak{d}C^\infty(\mathcal{B}) \tag{2.1.5}$$

to the derivation module of the $\mathbb{R}$-ring $C^\infty(\mathcal{B})$.

Let us consider the Chevalley–Eilenberg complex of the real Lie algebra $\mathcal{T}_1(\mathcal{B})$ with coefficients in $C^\infty(\mathcal{B})$ and its subcomplex $\mathcal{O}^*[\mathcal{T}_1(\mathcal{B})]$ of $C^\infty(\mathcal{B})$-multilinear skew-symmetric

maps by analogy with the complex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ in Section 1.4 [41]. This subcomplex is a differential calculus over a $\mathbb{R}$-ring $C^\infty(\mathcal{B})$ where the Chevalley–Eilenberg coboundary operator d (1.4.6) and the product (1.4.9) read

$$d\phi(\vartheta_0,\ldots,\vartheta_r) = \sum_{i=0}^{r}(-1)^i\partial_{\vartheta_i}(\phi(\vartheta_0,\ldots,\widehat{\vartheta_i},\ldots,\vartheta_r)) + \tag{2.1.6}$$
$$\sum_{i<j}(-1)^{i+j}\phi([\vartheta_i,\vartheta_j],\vartheta_0,\ldots,\widehat{\vartheta}_i,\ldots,\widehat{\vartheta}_j,\ldots,\vartheta_k),$$
$$\phi\wedge\phi'(\vartheta_1,...,\vartheta_{r+s}) = \tag{2.1.7}$$
$$\sum_{i_1<\cdots<i_r;j_1<\cdots<j_s}\mathrm{sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}\phi(\vartheta_{i_1},\ldots,\vartheta_{i_r})\phi'(\vartheta_{j_1},\ldots,\vartheta_{j_s}),$$
$$\phi\in\mathcal{O}^r[\mathcal{T}_1(\mathcal{B})],\qquad \phi'\in\mathcal{O}^s[\mathcal{T}_1(\mathcal{B})],\qquad \vartheta_i\in\mathcal{T}_1(\mathcal{B}).$$

There are the familiar relations

$$\vartheta\rfloor df = \partial_\vartheta f,\qquad f\in C^\infty(\mathcal{B}),\qquad \vartheta\in\mathcal{T}_1(\mathcal{B}),$$
$$d(\phi\wedge\phi') = d\phi\wedge\phi' + (-1)^{|\phi|}\phi\wedge d\phi',\qquad \phi,\phi'\in\mathcal{O}^*[\mathcal{T}_1(\mathcal{B})].$$

The differential calculus $\mathcal{O}^*[\mathcal{T}_1(\mathcal{B})]$ contains the following subcomplex. Let $\mathcal{O}^1(\mathcal{B})$ be the $C^\infty(\mathcal{B})$-module of global sections of the cotangent bundle $T^*\mathcal{B}$ of $\mathcal{B}$. Obviously, there is its monomorphism

$$\mathcal{O}^1(\mathcal{B})\to\mathfrak{d}C^\infty(\mathcal{B})^* \tag{2.1.8}$$

to the dual of the derivation module $\mathfrak{d}C^\infty(\mathcal{B})$. Furthermore, let $\overset{r}{\wedge}T^*\mathcal{B}$ be the r-degree exterior product of the cotangent bundle $T^*\mathcal{B}$ and $\mathcal{O}^r(\mathcal{B})$ the $C^\infty(\mathcal{B})$-module of its sections. Let $\mathcal{O}^*(\mathcal{B})$ be the direct sum of $C^\infty(\mathcal{B})$-modules $\mathcal{O}^r(\mathcal{B})$, $r\in\mathbb{N}$, where we put $\mathcal{O}^0(\mathcal{B}) = C^\infty(\mathcal{B})$. Elements of $\mathcal{O}^*(\mathcal{B})$ are obviously $C^\infty(\mathcal{B})$-multilinear skew-symmetric maps of $\mathcal{T}_1(\mathcal{B})$ to $C^\infty(\mathcal{B})$. Therefore, the Chevalley–Eilenberg differential d (2.1.6) and the exterior product (2.1.7) of elements of $\mathcal{O}^*(\mathcal{B})$ are well defined. Moreover, one can show that $d\phi$ and $\phi\wedge\phi'$, $\phi,\phi'\in\mathcal{O}^*(\mathcal{B})$, are also elements of $\mathcal{O}^*(\mathcal{B})$. Thus, $\mathcal{O}^*(\mathcal{B})$ is a differential graded commutative algebra, called the algebra of *exterior forms* on a Banach manifold $\mathcal{B}$.

At the same time, one can consider Chevalley–Eilenberg differential calculus $\mathcal{O}^*[\mathfrak{d}C^\infty(\mathcal{B})]$ over the $\mathbb{R}$-ring $C^\infty(\mathcal{B})$. Because of the monomorphism (2.1.5), we have the homomorphism of $C^\infty(\mathcal{B})$-modules

$$\mathcal{O}^1[\mathfrak{d}C^\infty(\mathcal{B})] = \mathfrak{d}C^\infty(\mathcal{B})^*\to\mathcal{T}_1(\mathcal{B})^* = \mathcal{O}^1[\mathcal{T}_1(\mathcal{B})]\leftarrow\mathcal{O}^1(\mathcal{B}). \tag{2.1.9}$$

It follows that the differential calculi $\mathcal{O}^*[\mathcal{T}_1(\mathcal{B})]$, $\mathcal{O}^*(\mathcal{B})$ and $\mathcal{O}^1[\mathfrak{d}C^\infty(\mathcal{B})]$ over the $\mathbb{R}$-ring $C^\infty(\mathcal{B})$ are not mutually isomorphic in general. However, it is readily observed that the minimal differential calculi in $\mathcal{O}^*[\mathcal{T}_1(\mathcal{B})]$ and $\mathcal{O}^*(\mathcal{B})$ coincide with the minimal Chevalley–Eilenberg differential calculus $\mathcal{O}^*C^\infty(\mathcal{B})$ over the $\mathbb{R}$-ring $C^\infty(\mathcal{B})$ because they

are generated by the elements df, $f \in C^\infty(\mathcal{B})$, where d is the restriction (2.1.6) to $\mathcal{T}_1(\mathcal{B})$ of the Chevalley–Eilenberg coboundary operator (1.4.6).

A *connection* on a Banach manifold $\mathcal{B}$ is defined as a connection on the $C^\infty(\mathcal{B})$-module $\mathcal{T}_1(\mathcal{B})$ [41, 86]. In accordance with Definition 1.3.2, it is an $\mathbb{R}$-module morphism

$$\nabla : \mathcal{T}_1(\mathcal{B}) \to \mathcal{O}^1 C^\infty(\mathcal{B}) \otimes \mathcal{T}_1(\mathcal{B}),$$

which obeys the Leibniz rule

$$\nabla(f\vartheta) = df \otimes \vartheta + f\nabla(\vartheta), \qquad f \in C^\infty(\mathcal{B}), \qquad \vartheta \in \mathcal{T}_1(\mathcal{B}). \tag{2.1.10}$$

In view of the inclusions,

$$\mathcal{O}^1 C^\infty(\mathcal{B}) \subset \mathcal{O}^1(\mathcal{B}) \subset \mathcal{T}_1(\mathcal{B})^*, \qquad \mathcal{T}_1(\mathcal{B}) \subset \mathcal{T}_1(\mathcal{B})^{**} \subset \mathcal{O}^1(\mathcal{B})^*,$$

it is however convenient to define a connection on a Banach manifold as an $\mathbb{R}$-module morphism

$$\nabla : \mathcal{T}_1(\mathcal{B}) \to \mathcal{O}^1(\mathcal{B}) \otimes \mathcal{T}_1(\mathcal{B}), \tag{2.1.11}$$

which obeys the Leibniz rule (2.1.10).

2.2 Geometry of Hilbert manifolds

Let us turn now to Hilbert manifolds. These are particular Banach manifolds modelled on complex Hilbert spaces, which are assumed to be separable (see Remark 2.1.1).

Remark 2.2.1: We refer the reader to [55] for the theory of real Hilbert and (infinite-dimensional) Riemannian manifolds. A real Hilbert manifold is a Banach manifold $\mathcal{B}$ modelled on a real Hilbert space V. It is assumed to be connected Hausdorff and paracompact space admitting the partition of unity by smooth functions (this is the case of a separable V). In infinite-dimensional geometry, the most of local results follow from general arguments analogous to those in the finite-dimensional case. The global theory of real Hilbert manifolds is more intricate. $\diamondsuit$

A complex Hilbert space $(E, \langle .|.\rangle)$ can be seen as a real Hilbert space

$$E \ni v \mapsto v_\mathbb{R} \in E_\mathbb{R}, \qquad (v_\mathbb{R}, v'_\mathbb{R}) = \mathrm{Re}\,\langle v|v'\rangle,$$

equipped with the complex structure $Jv_\mathbb{R} = (iv)_\mathbb{R}$. We have

$$(Jv_\mathbb{R}, Jv'_\mathbb{R}) = (v_\mathbb{R}, v'_\mathbb{R}), \qquad (Jv_\mathbb{R}, v'_\mathbb{R}) = \mathrm{Im}\,(v'_\mathbb{R}, v_\mathbb{R}).$$

Let $E_\mathbb{C} = \mathbb{C} \otimes E_\mathbb{R}$ denote the complexification of $E_\mathbb{R}$ provided with the Hermitian form $\langle .|.\rangle_\mathbb{C}$. The complex structure J on $E_\mathbb{R}$ is naturally extended to $E_\mathbb{C}$ by letting $J \circ i = i \circ J$. Then $E_\mathbb{C}$ is split into the two complex subspaces

$$\begin{aligned} &E_\mathbb{C} = E^{1,0} \oplus E^{0,1}, \\ &E^{1,0} = \{v_\mathbb{R} - iJv_\mathbb{R} \,:\, v_\mathbb{R} \in E_\mathbb{R}\}, \\ &E^{0,1} = \{v_\mathbb{R} + iJv_\mathbb{R} \,:\, v_\mathbb{R} \in E_\mathbb{R}\}, \end{aligned} \tag{2.2.1}$$

which are mutually orthogonal with respect to the Hermitian form $\langle .|.\rangle_\mathbb{C}$. Since

$$\langle v_\mathbb{R} - iJv_\mathbb{R})|v'_\mathbb{R} - iJv'_\mathbb{R}\rangle = 2\langle v|v'\rangle, \qquad \langle v_\mathbb{R} + iJv_\mathbb{R}|v'_\mathbb{R} + iJv'_\mathbb{R}\rangle = 2\langle v'|v\rangle,$$

there are the following linear and antilinear isometric bijections

$$\begin{aligned} &E \ni v \mapsto v_\mathbb{R} \to \frac{1}{\sqrt{2}}(v_\mathbb{R} - iJv_\mathbb{R}) \in E^{1,0}, \\ &E \ni v \mapsto v_\mathbb{R} \to \frac{1}{\sqrt{2}}(v_\mathbb{R} + iJv_\mathbb{R}) \in E^{0,1}. \end{aligned}$$

They make $E^{1,0}$ and $E^{0,1}$ isomorphic to the Hilbert space E and the dual Hilbert space $\overline{E}$, respectively. Hence, the decomposition (2.2.1) takes the form

$$E_\mathbb{C} = E \oplus \overline{E}. \tag{2.2.2}$$

The complex structure J on the direct sum (2.2.2) reads

$$J : E \oplus \overline{E} \ni v + \overline{u} \mapsto iv - i\overline{u} \in E \oplus \overline{E}, \tag{2.2.3}$$

where E and $\overline{E}$ are the (holomorphic and antiholomorphic) eigenspaces of J characterized by the eigenvalues i and $-i$, respectively.

Let f be a function (not necessarily linear) from a Hilbert space E to a Hilbert space H. It is said to be *differentiable* if the corresponding function $f_\mathbb{R}$ between the real Banach spaces $E_\mathbb{R}$ and $H_\mathbb{R}$ is differentiable. Let $df_\mathbb{R}(z)$, $z \in E_\mathbb{R}$, be the differential (2.1.2) of $f_\mathbb{R}$ on $E_\mathbb{R}$ which is a continuous linear morphism

$$E_\mathbb{R} \ni v_\mathbb{R} \mapsto df_\mathbb{R}(z)v_\mathbb{R} \in H_\mathbb{R}$$

between real topological vector spaces $E_\mathbb{R}$ and $H_\mathbb{R}$. This morphism is naturally extended to the $\mathbb{C}$-linear morphism

$$E_\mathbb{C} \ni v_\mathbb{C} \mapsto df_\mathbb{R}(z)v_\mathbb{C} \in H_\mathbb{C} \tag{2.2.4}$$

between the complexifications of $E_\mathbb{R}$ and $H_\mathbb{R}$. In view of the decomposition (2.2.2), one can introduce the $\mathbb{C}$-linear maps

$$\partial f_\mathbb{R}(z)(v + \overline{u}) = df_\mathbb{R}(z)v, \qquad \overline{\partial} f(z)(v + \overline{u}) = df_\mathbb{R}(z)\overline{u}$$

from $E\oplus\overline{E}$ to $H_{\mathbb{C}}$ such that

$$df_{\mathbb{R}}(z)v_{\mathbb{C}} = df_{\mathbb{R}}(z)(v+\overline{u}) = \partial f_{\mathbb{R}}(z)v + \overline{\partial} f_{\mathbb{R}}(z)\overline{u}.$$

Let us split

$$f_{\mathbb{R}}(z) = f(z) + \overline{f}(z)$$

in accordance with the decomposition $H_{\mathbb{C}} = H\oplus\overline{H}$. Then the morphism (2.2.4) takes the form

$$df_{\mathbb{R}}(z)(v+\overline{u}) = \partial f(z)v + \overline{\partial} f(z)\overline{u} + \partial\overline{f}(z)v + \overline{\partial}\,\overline{f}(z)\overline{u}, \tag{2.2.5}$$

where $\partial\overline{f} = \overline{\overline{\partial} f}$, $\overline{\partial}\,\overline{f} = \overline{\partial f}$. A function $f: E\to H$ is said to be *holomorphic* (resp. *antiholomorphic*) if it is differentiable and $\overline{\partial} f(z) = 0$ (resp. $\partial f(z) = 0$) for all $z\in E$. A holomorphic function is smooth, and is given by the Taylor series. If f is a holomorphic function, then the morphism (2.2.5) is split into the sum

$$df_{\mathbb{R}}(z)(v+\overline{u}) = \partial f(z)v + \overline{\partial}\,\overline{f}(z)\overline{u}$$

of morphisms $E\to H$ and $\overline{E}\to\overline{H}$.

Example 2.2.2: Let f be a complex function on a Hilbert space E. Then

$$f_{\mathbb{R}} = (\mathrm{Re}\, f, \mathrm{Im}\, f)$$

is a map of E to $\mathbb{R}^2$. The differential $df_{\mathbb{R}}(z)$, $z\in E$, of $f_{\mathbb{R}}$ yields the complex linear morphism

$$E\oplus\overline{E}\ni v_{\mathbb{C}}\mapsto (d\mathrm{Re}\, f(z)v_{\mathbb{C}}, d\mathrm{Im}\, f(z)v_{\mathbb{C}})\mapsto d(\mathrm{Re}\, f + i\mathrm{Im}\, f)(z)v_{\mathbb{C}}\in\mathbb{C},$$

which is regarded as a differential $df(z)$ of a complex function f on a Hilbert space E. $\diamondsuit$

A *Hilbert manifold* $\mathcal{P}$ modelled on a Hilbert space E is defined as a real Banach manifold modelled on the Banach space $E_{\mathbb{R}}$ which admits an atlas $\{(U_\iota,\phi_\iota)\}$ with holomorphic transition functions $\phi_\zeta\phi_\iota^{-1}$. Let $CT\mathcal{P}$ denote the *complexified tangent bundle* of a Hilbert manifold $\mathcal{P}$. In view of the decomposition (2.2.2), each fibre $CT_z\mathcal{P}$, $z\in\mathcal{P}$, of $CT\mathcal{P}$ is split into the direct sum

$$CT_z\mathcal{P} = T_z\mathcal{P}\oplus\overline{T}_z\mathcal{P}$$

of subspaces $T_z\mathcal{P}$ and $\overline{T}_z\mathcal{P}$, which are topological complex vector spaces isomorphic to the Hilbert space E and the dual Hilbert space $\overline{E}$, respectively. The spaces $CT_z\mathcal{P}$, $T_z\mathcal{P}$ and $\overline{T}_z\mathcal{P}$ are respectively called the *complex*, *holomorphic* and *antiholomorphic tangent spaces* to a Hilbert manifold $\mathcal{P}$ at a point $z\in\mathcal{P}$. Since transition functions of a Hilbert manifold are holomorphic, the complex tangent bundle $CT\mathcal{P}$ is split into a sum

$$CT\mathcal{P} = T\mathcal{P}\oplus\overline{T}\mathcal{P}$$

of *holomorphic* and *antiholomorphic* subbundles, together with the antilinear bundle automorphism

$$T\mathcal{P} \oplus \overline{T}\mathcal{P} \ni v + \overline{u} \mapsto \overline{v} + u \in T\mathcal{P} \oplus \overline{T}\mathcal{P}$$

and the complex structure

$$J : T\mathcal{P} \oplus \overline{T}\mathcal{P} \ni v + \overline{u} \mapsto iv - i\overline{u} \in T\mathcal{P} \oplus \overline{T}\mathcal{P}. \qquad (2.2.6)$$

Sections of the complex tangent bundle $CT\mathcal{P} \to \mathcal{P}$ are called *complex vector fields* on a Hilbert manifold $\mathcal{P}$. They constitute the locally free module $C\mathcal{T}_1(\mathcal{P})$ over the ring $\mathbb{C}^\infty(\mathcal{P})$ of smooth complex functions on $\mathcal{P}$. Every complex vector field $\vartheta + \overline{v}$ on $\mathcal{P}$ yields a derivation

$$f(z) \to df(z)(\vartheta + \upsilon) = \partial f(z)\vartheta(z) + \overline{\partial} f(z)\upsilon(z), \qquad f \in \mathbb{C}^\infty(\mathcal{P}), \quad z \in \mathcal{P},$$

of the $\mathbb{C}$-ring $\mathbb{C}^\infty(\mathcal{P})$.

The (topological) dual of the complex tangent bundle $CT\mathcal{P}$ is the *complex cotangent bundle* $CT^*\mathcal{P}$ of $\mathcal{P}$. Its fibres $CT^*_z\mathcal{P}$, $z \in \mathcal{P}$, are topological complex vector spaces isomorphic to $E \oplus \overline{E}$. Since Hilbert spaces are reflexive, the complex tangent bundle $CT\mathcal{P}$ is the dual of $CT^*\mathcal{P}$. The complex cotangent bundle $CT^*\mathcal{P}$ is split into the sum

$$CT^*\mathcal{P} = T^*\mathcal{P} \oplus \overline{T}^*\mathcal{P} \qquad (2.2.7)$$

of *holomorphic* and *antiholomorphic* subbundles, which are the annihilators of antiholomorphic and holomorphic tangent bundles $\overline{T}\mathcal{P}$ and $T\mathcal{P}$, respectively. Accordingly, $CT^*\mathcal{P}$ is provided with the complex structure J via the relation

$$\langle v, Jw \rangle = \langle Jv, w \rangle, \qquad v \in CT_z\mathcal{P}, \qquad w \in CT^*_z\mathcal{P}, \qquad z \in \mathcal{P}.$$

Sections of the complex cotangent bundle $CT^*\mathcal{P} \to \mathcal{P}$ constitute a locally free $\mathbb{C}^\infty(\mathcal{P})$-module $\mathcal{O}^1(\mathcal{P})$. It is the $\mathbb{C}^\infty(\mathcal{P})$-dual

$$\mathcal{O}^1(\mathcal{P}) = C\mathcal{T}_1(\mathcal{P})^* \qquad (2.2.8)$$

of the module $C\mathcal{T}_1(\mathcal{P})$ of complex vector fields on $\mathcal{P}$, and *vice versa*.

Similarly to the case of a Banach manifold, let us consider the differential calculi $\mathcal{O}^*[\mathcal{T}_1(\mathcal{P})]$, $\mathcal{O}^*(\mathcal{P})$ (further denoted by $\mathcal{C}^*(\mathcal{P})$) and $\mathcal{O}^1[\mathfrak{d}C^\infty(\mathcal{P})]$ over the $\mathbb{C}$-ring $\mathbb{C}^\infty(\mathcal{P})$. Due to the isomorphism (2.2.8), $\mathcal{O}^*[\mathcal{T}_1(\mathcal{P})]$ is isomorphic to $\mathcal{C}^*(\mathcal{P})$, whose elements are called *exterior form!complex* on a Hilbert manifold $\mathcal{P}$. The exterior differential d on these forms is the Chevalley–Eilenberg coboundary operator

$$d\phi(\vartheta_0, \ldots, \vartheta_k) = \sum_{i=0}^{k} (-1)^i d\phi(\vartheta_0, \ldots, \widehat{\vartheta_i}, \ldots, \vartheta_k)\vartheta_i + \qquad (2.2.9)$$
$$\sum_{i<j} (-1)^{i+j} \phi([\vartheta_i, \vartheta_j], \vartheta_0, \ldots, \widehat{\vartheta}_i, \ldots, \widehat{\vartheta}_j, \ldots, \vartheta_k), \qquad \vartheta_i \in C\mathcal{T}_1(\mathcal{P}).$$

In view of the splitting (2.2.7), the differential graded algebra $\mathcal{C}^*(\mathcal{P})$ admits the decomposition

$$\mathcal{C}^*(\mathcal{P}) = \bigoplus_{p,q=0} \mathcal{C}^{p,q}(\mathcal{P})$$

into subspaces $\mathcal{C}^{p,q}(\mathcal{P})$ of *p-holomorphic* and *q-antiholomorphic* forms. Accordingly, the exterior differential d on $\mathcal{C}^*(\mathcal{P})$ is split into a sum $d = \partial + \overline{\partial}$ of holomorphic and antiholomorphic differentials

$$\partial : \mathcal{C}^{p,q}(\mathcal{P}) \to \mathcal{C}^{p+1,q}(\mathcal{P}), \qquad \overline{\partial} : \mathcal{C}^{p,q}(\mathcal{P}) \to \mathcal{C}^{p,q+1}(\mathcal{P}),$$
$$\partial \circ \partial = 0, \qquad \overline{\partial} \circ \overline{\partial} = 0, \qquad \partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0.$$

A *Hermitian metric* on a Hilbert manifold $\mathcal{P}$ is defined as a complex bilinear form g on fibres of the complex tangent bundle $CT\mathcal{P}$ which obeys the following conditions:

• g is a smooth section of the tensor bundle $CT^*\mathcal{P} \otimes CT^*\mathcal{P} \to \mathcal{P}$;

• $g(\vartheta_z, \vartheta'_z) = 0$ if complex tangent vectors $\vartheta_z, \vartheta'_z \in CT_z\mathcal{P}$ are simultaneously holomorphic or antiholomorphic;

• $g(\vartheta_z, \overline{\vartheta}_z) > 0$ for any non-vanishing complex tangent vector $\vartheta_z \in CT_z\mathcal{P}$;

• the bilinear form $g(\vartheta_z, \vartheta'_z)$, $\vartheta_z, \vartheta'_z \in CT_z\mathcal{P}$, defines a norm topology on the complex tangent space $CT_z\mathcal{P}$ which is equivalent to its Hilbert space topology.

As an immediate consequence of this definition, we obtain

$$\overline{g(\vartheta_z, \vartheta'_z)} = g(\overline{\vartheta}_z, \overline{\vartheta}'_z), \qquad g(J\vartheta_z, J\vartheta'_z) = g(\vartheta_z, \vartheta'_z).$$

A Hermitian metric exists, e.g., on paracompact Hilbert manifolds modelled on separable Hilbert spaces.

The above mentioned properties of a Hermitian metric on a Hilbert manifold are similar to properties of a Hermitian metric on a finite-dimensional complex manifold. Therefore, one can think of the pair $(\mathcal{P}, g)$ as being an infinite-dimensional *Hermitian manifold*.

A Hermitian manifold $(\mathcal{P}, g)$ is endowed with a non-degenerate exterior two-form

$$\Omega(\vartheta_z, \vartheta'_z) = g(J\vartheta_z, \vartheta'_z), \qquad \vartheta_z, \vartheta'_z \in CT_z\mathcal{P}, \qquad z \in \mathcal{P}, \tag{2.2.10}$$

called the *fundamental form* of the Hermitian metric g. This form satisfies the relations

$$\overline{\Omega(\vartheta_z, \vartheta'_z)} = \Omega(\overline{\vartheta}_z, \overline{\vartheta}'_z), \qquad \Omega(J_z\vartheta_z, J_z\vartheta'_z) = \Omega(\vartheta_z, \vartheta'_z).$$

If Ω (2.2.10) is a closed (i.e., symplectic) form, the Hermitian metric g is called a *Kähler metric* and Ω a *Kähler form*. Accordingly, $(\mathcal{P}, g, \Omega)$ is said to be an infinite-dimensional *Kähler manifold*.

By analogy with the case of a Banach manifold, we modify Definition 1.3.2 and define a *connection* ∇ on a Hilbert manifold $\mathcal{P}$ as a $\mathbb{C}$-module morphism

$$\nabla : C\mathcal{T}_1(\mathcal{P}) \to \mathcal{C}^1(\mathcal{P}) \otimes C\mathcal{T}_1(\mathcal{P}),$$

which obeys the Leibniz rule

$$\nabla(f\vartheta) = df\otimes\vartheta + f\nabla(\vartheta), \qquad f\in\mathbb{C}^\infty(\mathcal{P}), \qquad \vartheta\in C\mathcal{T}_1(\mathcal{P}).$$

Similarly, a connection is introduced on any $\mathbb{C}^\infty(\mathcal{P})$-module, e.g., on sections of tensor bundles over a Hilbert manifold $\mathcal{P}$. Let D and $\overline{D}$ denote the holomorphic and anti-holomorphic parts of ∇, and let $\nabla_\vartheta = \vartheta\rfloor\nabla$, D_ϑ and $\overline{D}_\vartheta$ be the corresponding covariant derivatives along a complex vector field ϑ on $\mathcal{P}$. For any complex vector field $\vartheta = \nu + \overline{\upsilon}$ on $\mathcal{P}$, we have the relations

$$D_\vartheta = \nabla_\nu, \qquad \overline{D}_\vartheta = \overline{\nabla}_{\overline{\upsilon}}, \qquad D_{J\vartheta} = iD_\vartheta, \qquad \overline{D}_{J\vartheta} = -i\overline{D}_\vartheta.$$

PROPOSITION 2.2.1: Given a Kähler manifold $(\mathcal{P}, g)$, there always exists a *metric connection* on $\mathcal{P}$ such that

$$\nabla g = 0, \qquad \nabla\Omega = 0, \qquad \nabla J = 0,$$

where J is regarded as a section of the tensor bundle $CT^*\mathcal{P}\otimes CT\mathcal{P}$. □

Example 2.2.3: If $\mathcal{P} = E$ is a Hilbert space, then

$$CT\mathcal{P} = E\times(E\oplus\overline{E}).$$

A Hermitian form $\langle .|.\rangle$ on E defines the constant Hermitian metric

$$g : (E\oplus\overline{E})\times(E\oplus\overline{E})\to\mathbb{C},$$
$$g(\vartheta,\vartheta') = \langle v|u'\rangle + \langle v'|u\rangle, \qquad \vartheta = v+\overline{u}, \quad \vartheta' = v'+\overline{u}', \tag{2.2.11}$$

on $\mathcal{P} = E$. The associated fundamental form (2.2.10) reads

$$\Omega(\vartheta,\vartheta') = i\langle v|u'\rangle - i\langle v'|u\rangle. \tag{2.2.12}$$

It is constant on E. Therefore, $d\Omega = 0$ and g (2.2.11) is a Kähler metric. The metric connection on E is trivial, i.e., $\nabla = d$, $D = \partial$, $\overline{D} = \overline{\partial}$. ♢

Example 2.2.4: Given a Hilbert space E, *a projective Hilbert space* PE is made up by the complex one-dimensional subspaces (i.e., complex rays) of E. This is a Hilbert manifold with the following standard atlas. For any non-zero element $x\in E$, let us denote by x a point of PE such that $x\in\mathrm{x}$. Then each normalized element $h\in E$, $||h|| = 1$, defines a chart (U_h, ϕ_h) of the projective Hilbert space PE such that

$$U_h = \{\mathrm{x}\in PE \ : \ \langle x|h\rangle \neq 0\}, \qquad \phi_h(\mathrm{x}) = \frac{x}{\langle x|h\rangle} - h. \tag{2.2.13}$$

The image of U_h in the Hilbert space E is the one-codimensional closed (Hilbert) subspace

$$E_h = \{z\in E \ : \ \langle z|h\rangle = 0\}, \tag{2.2.14}$$

where $z(\mathrm{x})+h\in\mathrm{x}$. In particular, given a point $\mathrm{x}\in PE$, one can choose the centered chart E_h, $h\in\mathrm{x}$, such that $\phi_h(\mathrm{x})=0$. Hilbert spaces E_h and $E_{h'}$ associated to different charts U_h and $U_{h'}$ are isomorphic. The transition function between them is a holomorphic function

$$z'(\mathrm{x})=\frac{z(\mathrm{x})+h}{\langle z(\mathrm{x})+h|h'\rangle}-h', \qquad \mathrm{x}\in U_h\cap U_{h'}, \tag{2.2.15}$$

from $\phi_h(U_h\cap U_{h'})\subset E_h$ to $\phi_{h'}(U_h\cap U_{h'})\subset E_{h'}$. The set of the charts $\{(U_h,\phi_h)\}$ with the transition functions (2.2.15) provides a holomorphic atlas of the projective Hilbert space PE. The corresponding coordinate transformations for the tangent vectors to PE at $\mathrm{x}\in PE$ reads

$$v'=\frac{1}{\langle x|h'\rangle}[\langle x|h\rangle v-x\langle v|h\rangle]. \tag{2.2.16}$$

The projective Hilbert space PE is homeomorphic to the quotient of the unitary group $U(E)$ equipped with the normed operator topology by the stabilizer of a ray of E. It is connected and simply connected [21].

The projective Hilbert space PE admits a unique Hermitian metric g such that the corresponding distance function on PE is

$$\rho(\mathrm{x},\mathrm{x}')=\sqrt{2}\arccos(|\langle x|x'\rangle|), \tag{2.2.17}$$

where x,x' are normalized elements of E. It is a Kähler metric, called the *Fubini–Studi metric*. Given a coordinate chart (U_h,ϕ_h), this metric reads

$$g_{\mathrm{FS}}(\vartheta_1,\vartheta_2)=\frac{\langle v_1|u_2\rangle+\langle v_2|u_1\rangle}{1+\|z\|^2}-\frac{\langle z|u_2\rangle\langle v_1|z\rangle+\langle z|u_1\rangle\langle v_2|z\rangle}{(1+\|z\|^2)^2}, \quad z\in E_h, \tag{2.2.18}$$

for any complex tangent vectors $\vartheta_1=v_1+\overline{u}_1$ and $\vartheta_2=v_2+\overline{u}_2$ in CT_zPE. The corresponding Kähler form is given by the expression

$$\Omega_{\mathrm{FS}}(\vartheta_1,\vartheta_2)=i\frac{\langle v_1|u_2\rangle-\langle v_2|u_1\rangle}{1+\|z\|^2}-i\frac{\langle z|u_2\rangle\langle v_1|z\rangle-\langle z|u_1\rangle\langle v_2|z\rangle}{(1+\|z\|^2)^2}. \tag{2.2.19}$$

It is readily justified that the expressions (2.2.18) – (2.2.19) are preserved under the transition functions (2.2.15) – (2.2.16). Written in the coordinate chart centered at a point $z(\mathrm{x})=0$, these expressions come to the expressions (2.2.11) and (2.2.12), respectively. $\diamondsuit$

2.3 Hilbert and C^*-algebra bundles

Due to the inverse mapping Theorem 2.1.1 for Banach manifolds, fibred Banach manifolds similarly to the finite-dimensional smooth ones can be defined as surjective submersions of Banach manifold onto Banach manifold. Locally trivial fibred Banach manifolds are called (smooth) *Banach vector bundle*. They are exemplified by vector (e.g., tangent and cotangent) bundles over Banach manifolds in Section 2.1. Here, we restrict our

consideration to particular Banach vector bundles whose fibres are C^*-algebras (seen as Banach spaces) and Hilbert spaces, but the base is a finite-dimensional smooth manifold.

Note that sections of a Banach vector bundle $\mathcal{B} \to Q$ over a smooth finite-dimensional manifold Q constitute a locally free $C^\infty(Q)$-module $\mathcal{B}(Q)$. Following the proof of Proposition 1.7.2, one can show that it is a projective $C^\infty(Q)$-module. In a general setting, we therefore can consider projective locally free $C^\infty(Q)$-modules, locally generated by a Banach space. In contrast with the case of projective $C^\infty(X)$ modules of finite rank, such a module need not be a module of sections of some Banach vector bundle.

Let $\mathcal{C} \to Q$ be a locally trivial topological fibre bundle over a finite-dimensional smooth real manifold Q whose typical fibre is a C^*-algebra A, regarded as a real Banach space, and whose transition functions are smooth. Namely, given two trivializations charts (U_1, ψ_1) and (U_2, ψ_2) of $\mathcal{C}$, we have the smooth morphism of Banach manifolds

$$\psi_1 \circ \psi_2^{-1} : U_1 \cap U_2 \times A \to U_1 \cap U_2 \times A,$$

where

$$\psi_1 \circ \psi_2^{-1}|_{q \in U_1 \cap U_2}$$

is an automorphism of A. We agree to call $\mathcal{C} \to Q$ a *bundle of C^*-algebras.* It is a Banach vector bundle. The $C^\infty(Q)$-module $\mathcal{C}(Q)$ of smooth sections of this fibre bundle is a unital involutive algebra with respect to fibrewise operations. Let us consider a subalgebra $A(Q) \subset \mathcal{C}(Q)$ which consists of sections of $\mathcal{C} \to Q$ vanishing at infinity on Q. It is provided with the norm

$$||\alpha|| = \sup_{q \in Q} ||\alpha(q)|| < \infty, \qquad \alpha \in A(Q), \tag{2.3.1}$$

but fails to be complete. Nevertheless, one extends $A(Q)$ to a C^*-algebra of continuous sections of $\mathcal{C} \to Q$ vanishing at infinity on a locally compact space Q.

Remark 2.3.1: Let $\mathcal{C} \to Q$ be a topological bundle of C^*-algebras over a locally compact topological space Q, and let $\mathcal{C}^0(Q)$ denote the involutive algebra of its continuous sections. This algebra exemplifies a locally trivial continuous *field of C^*-algebras* in [31]. Its subalgebra $A^0(Q)$ of sections vanishing at infinity on Q is a C^*-algebra with respect to the norm (2.3.1). It is called a *C^*-algebra defined by a continuous field of C^*-algebras.* There are several important examples of C^*-algebras of this type. For instance, any commutative C^*-algebra is isomorphic to the algebra of continuous complex functions vanishing at infinity on its spectrum. $\diamondsuit$

Remark 2.3.2: One can consider a locally trivial bundle of C^*-algebras $\mathcal{C} \to Q$ as a fibre bundle with the structure topological group $\mathrm{Aut}(A)$ of automorphisms of A. If a fibre bundle $\mathcal{C}$ is smooth, this group is necessarily provided with a normed operator topology. $\diamondsuit$

Hilbert bundles over a smooth manifold are similarly defined. Let $\mathcal{E} \to Q$ be a locally trivial topological fibre bundle over a finite-dimensional smooth real manifold Q

whose typical fibre is a Hilbert space E, regarded as a real Banach space, and whose transition functions are smooth functions taking their values in the unitary group $U(E)$ equipped with the normed operator topology. We agree to call $\mathcal{E} \to Q$ a *Hilbert bundle*. It is a Banach vector bundle. Smooth sections of $\mathcal{E} \to Q$ constitute a $C^\infty(Q)$-module $\mathcal{E}(Q)$, called a *Hilbert module*. Continuous sections of $\mathcal{E} \to Q$ constitute a locally trivial continuous field of Hilbert spaces [31].

There are the following relations between bundles of C^*-algebras and Hilbert bundles.

Let $T(E) \subset B(E)$ be the C^*-algebra of compact (completely continuous) operators in a Hilbert space E. It is called an *elementary C^*-algebra*. Every automorphism ϕ of E yields the corresponding automorphism

$$T(E) \to \phi T(E)\phi^{-1}$$

of the C^*-algebra $T(E)$. Therefore, given a Hilbert bundle $\mathcal{E} \to Q$ with transition functions

$$E \to \rho_{\iota\zeta}(q)E, \qquad q \in U_\iota \cap U_\zeta,$$

over a cover $\{U_\iota\}$ of Q, we have the associated locally trivial bundle of elementary C^*-algebras $T(E)$ with transition functions

$$T(E) \to \rho_{\alpha\beta}(q)T(E)(\rho_{\alpha\beta}(q))^{-1}, \qquad q \in U_\alpha \cap U_\beta, \tag{2.3.2}$$

which are proved to be continuous with respect to the normed operator topology on $T(E)$ [31]. The proof is based on the following facts.

• The set of degenerate operators (i.e., operators of finite rank) is dense in $T(E)$.

• Any operator of finite rank is a linear combination of operators

$$P_{\xi,eta} : \zeta \to \langle\zeta|\eta\rangle\xi, \qquad \xi, \eta, \zeta \in E,$$

and even the projectors P_ξ onto $\xi \in E$.

• Let $\xi_1, \ldots, \xi_{2n}$ be variable vectors of E. If ξ_i, $i = 1, \ldots, 2n$, converges to η_i (or, more generally, $\langle\xi_i|\xi_j\rangle$ converges to $\langle\eta_i|\eta_j\rangle$ for any i and j), then

$$P_{\xi_1,\xi_2} + \cdots + P_{\xi_{2n-1},\xi_{2n}}$$

uniformly converges to

$$P_{\eta_1,\eta_2} + \cdots + P_{\eta_{2n-1},\eta_{2n}}.$$

Note that, given a Hilbert bundle $\mathcal{E} \to Q$, the associated bundle of C^*-algebras $B(E)$ of bounded operators in E fails to be constructed in general because the transition functions (2.3.2) need not be continuous.

The opposite construction however meets a topological obstruction as follows [14, 15].

Let $\mathcal{C} \to Q$ be a bundle of C^*-algebras whose typical fibre is an elementary C^*-algebra $T(E)$ of compact operators in a Hilbert space E. One can think of $\mathcal{C} \to Q$ as being a topological fibre bundle with the structure group of automorphisms of $T(E)$. This is the projective unitary group $PU(E)$. With respect to the normed operator topology, the groups $U(E)$ and $PU(E)$ are the Banach Lie groups [45]. Moreover, $U(E)$ is contractible if a Hilbert space E is infinite-dimensional [52]. Let $(U_\alpha, \rho_{\alpha\beta})$ be an atlas of the fibre bundle $\mathcal{C} \to Q$ with $PU(E)$-valued transition functions $\rho_{\alpha\beta}$. These transition functions give rise to the maps

$$\overline{\rho}_{\alpha\beta} : U_\alpha \cap U_\beta \to U(E),$$

which however fail to be transition functions of a fibre bundle with the structure group $U(E)$ because they need not satisfy the cocycle condition. Their failure to be so is measured by the $U(1)$-valued cocycle

$$e_{al\beta\gamma} = \overline{g}_{\beta\gamma}\overline{g}^{-1}_{\alpha\gamma}\overline{g}_{\alpha\beta}.$$

This cocycle defines a class $[e]$ in the cohomology group $H^2(Q; U(1)_Q)$ of the manifold Q with coefficients in the sheaf $U(1)_Q$ of continuous maps of Q to $U(1)$. This cohomology class vanishes iff there exists a Hilbert bundle associated to $\mathcal{C}$. Let us consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \longrightarrow C^0_Q \xrightarrow{\gamma} U(1)_Q \to 0,$$

where C^0_Q is the sheaf of continuous real functions on Q and the morphism γ reads

$$\gamma : C^0_Q \ni f \mapsto \exp(2\pi i f) \in U(1)_Q.$$

This exact sequence yields the long exact sequence (5.3.16) of the sheaf cohomology groups

$$\begin{array}{l} 0 \to \mathbb{Z} \longrightarrow C^0_Q \longrightarrow U(1)_Q \longrightarrow H^1(Q;\mathbb{Z}) \longrightarrow \cdots \\ \quad H^p(Q;\mathbb{Z}) \longrightarrow H^p(Q;C^0_Q) \longrightarrow H^p(Q;U(1)_Q) \longrightarrow H^{p+1}(Q;\mathbb{Z}) \longrightarrow \cdots . \end{array}$$

Since the sheaf C^0_Q is fine and acyclic, we obtain at once from this exact sequence the isomorphism of cohomology groups

$$H^2(Q, U(1)_Q) = H^3(Q, \mathbb{Z}).$$

The image of $[e]$ in $H^3(Q, \mathbb{Z})$ is called the *Dixmier–Douady class* [31]. One can show that the negative $-[e]$ of the Dixmier–Douady class is the obstruction class of the lift of $PU(E)$-principal bundles to the $U(E)$-principal ones [15].

Thus, studying Hilbert and C^*-algebra bundles, we come to fibre bundles with unitary and projective unitary structure groups.

2.4 Connections on Hilbert and C^*-algebra bundles

There are different notions of a connection on Hilbert and C^*-algebra bundles which need not be obviously equivalent, unless bundles are finite-dimensional. These are connections on structure modules of sections, connections as a horizontal splitting and principal connections.

Given a bundle of C^*-algebras $\mathcal{C} \to Q$ with a typical fibre A over a smooth real manifold Q, the involutive algebra $\mathcal{C}(Q)$ of its smooth sections is a $C^\infty(Q)$-algebra. Therefore, one can introduce a connection on the fibre bundle $\mathcal{C} \to Q$ as a connection on the $C^\infty(Q)$-algebra $\mathcal{C}(Q)$. In accordance with Definition 1.3.4, such a *connection* assigns to each vector field τ on Q a symmetric derivation ∇_τ of the involutive algebra $\mathcal{C}(Q)$ which obeys the Leibniz rule

$$\nabla_\tau(f\alpha) = (\tau\rfloor df)\alpha + f\nabla_\tau\alpha, \qquad f \in \mathbb{C}^\infty(Q), \qquad \alpha \in \mathcal{C}(Q),$$

and the condition

$$\nabla_\tau\alpha^* = (\nabla_\tau\alpha)^*.$$

Let us recall that two such connections ∇_τ and ∇'_τ differ from each other in a derivation of the $C^\infty(Q)$-algebra $\mathcal{C}(Q)$. Then, given a trivialization chart

$$\mathcal{C}|_U \cong U \times A$$

of $\mathcal{C} \to Q$, a connection on $\mathcal{C}(Q)$ can be written in the form

$$\nabla_\tau = \tau^m(q)(\partial_m - \delta_m(q)), \qquad q \in U, \tag{2.4.1}$$

where (q^m) are local coordinates on Q and $\delta_m(q)$ for all $q \in U$ are symmetric bounded derivations of the C^*-algebra A.

Remark 2.4.1: A problem is that a C^*-algebra need not admit bounded derivations. For instance, no commutative algebra possesses bounded derivations. Moreover, a bounded derivation of a C^*-algebra is the infinitesimal generator of a uniformly continuous one-parameter group of automorphisms of this algebra [10]. However, strongly continuous groups of C^*-algebra automorphisms usually are considered in quantum models [41]. Therefore, one should assume that, in these models, the derivations $\delta_m(q)$ in the expression (2.4.1) are unbounded in general. This leads us to the notion of a *generalized connection* on bundles of C^*-algebras [4]. $\diamondsuit$

Let $\mathcal{E} \to Q$ be a Hilbert bundle with a typical fibre E and $\mathcal{E}(Q)$ the $C^\infty(Q)$-module of its smooth sections. Then a *connection on a Hilbert bundle* $\mathcal{E} \to Q$ is defined as a connection ∇ on the module $\mathcal{E}(Q)$. In accordance with Definition 1.6.5, such a connection assigns to each vector field τ on Q a first order differential operator ∇_τ on $\mathcal{E}(Q)$ which obeys both the Leibniz rule

$$\nabla_\tau(f\psi) = (\tau\rfloor df)\psi + f\nabla_\tau\psi, \qquad f \in \mathbb{C}^\infty(Q), \qquad \psi \in \mathcal{E}(Q),$$

and the additional condition

$$\langle(\nabla_\tau\psi)(q)|\psi(q)\rangle + \langle\psi(q)|(\nabla_\tau\psi)(q)\rangle = \tau(q)\rfloor d\langle\psi(q)|\psi(q)\rangle. \tag{2.4.2}$$

Given a trivialization chart $\mathcal{E}|_U \cong U \times E$ of $\mathcal{E} \to Q$, a connection on $\mathcal{E}(Q)$ reads

$$\nabla_\tau = \tau^m(q)(\partial_m + i\mathcal{H}_m(q)), \qquad q \in U, \tag{2.4.3}$$

where $\mathcal{H}_m(q)$ for all $q \in U$ are bounded self-adjoint operators in the Hilbert space E.

In a general setting, let $\mathcal{B} \to Q$ be a Banach vector bundle over a finite-dimensional smooth manifold Q and $\mathcal{B}(X)$ the locally free $C^\infty(Q)$-module of its smooth sections $s(q)$. By virtue of Definition 1.6.5, a connection on $\mathcal{B}(Q)$ assigns to each vector field τ on Q a first order differential operator ∇_τ on $\mathcal{B}(Q)$ which obeys the Leibniz rule

$$\nabla_\tau(fs) = (\tau\rfloor df)s + f\nabla_\tau s, \qquad f \in \mathbb{C}^\infty(Q), \qquad s \in \mathcal{B}(Q). \tag{2.4.4}$$

In accordance with Proposition 1.7.4, such a connection exists. Connections (2.4.1) and (2.4.3) exemplify connections on Banach vector bundles $\mathcal{C} \to Q$ and $\mathcal{E} \to Q$, but they obey additional conditions because these bundles possess additional structures of a C^*-algebra bundle and a Hilbert bundle, respectively. In particular, the connection (2.4.3) is a principal connection whose second term is an element of the Lie algebra of the unitary group $U(E)$.

In a different way, connection on a Banach vector bundle $\mathcal{B} \to Q$ can be defined as a splitting of the exact sequence

$$0 \to V\mathcal{B} \to T\mathcal{B} \to TQ \underset{Q}{\otimes} \mathcal{B} \to 0,$$

where $V\mathcal{B}$ denotes the vertical tangent bundle of $\mathcal{B} \to Q$. In the case of finite-dimensional vector bundles, both definitions are equivalent (Section 1.6). This equivalence is extended to the case of Banach vector bundles over a finite-dimensional base. We leave the proof of this fact outside the scope of our exposition because it involves the notion of jets of Banach fibre bundles.

2.5 Instantwise quantization

This Section addresses the evolution of such quantum systems which can be viewed as a parallel displacement along time.

It should be emphasized that, in quantum mechanics based on the Schrödinger and Heisenberg equations, the physical time plays the role of a classical parameter. Indeed, all relations between operators in quantum mechanics are simultaneous, while computation of mean values of operators in a quantum state does not imply integration over time. It follows that, at each instant $t \in \mathbb{R}$, there is an instantaneous quantum system characterized by some C^*-algebra A_t. Thus, we come to *instantwise quantization*. Let us

suppose that all instantaneous C^*-algebras A_t are isomorphic to some unital C^*-algebra A. Furthermore, let they constitute a locally trivial smooth bundle $\mathcal{C}$ of C^*-algebras over the time axis $\mathbb{R}$. Its typical fibre is A. This bundle of C^*-algebras is trivial, but need not admit a canonical trivialization in general. One can think of its different trivializations as being associated to different reference frames [41, 78].

Let us describe evolution of quantum systems in the framework of instantwise quantization. Given a bundle of C^*-algebras $\mathcal{C} \to \mathbb{R}$, this evolution can be regarded as a parallel displacement with respect to some connection on $\mathcal{C} \to \mathbb{R}$ [4, 75]. Following previous Section, we define ∇ as a connection on the involutive $C^\infty(\mathbb{R})$-algebra $\mathcal{C}(\mathbb{R})$ of smooth sections of $\mathcal{C} \to \mathbb{R}$. It assigns to the standard vector field ∂_t on $\mathbb{R}$ a symmetric derivation ∇_t of $\mathcal{C}(\mathbb{R})$ which obeys the Leibniz rule

$$\nabla_t(f\alpha) = \partial_t f\alpha + f\nabla_t\alpha, \qquad \alpha \in \mathcal{C}(\mathbb{R}), \qquad f \in C^\infty(\mathbb{R}),$$

and the condition

$$\nabla_t\alpha^* = (\nabla_t\alpha)^*.$$

Given a trivialization $\mathcal{C} = \mathbb{R} \times A$, a connection ∇_t reads

$$\nabla_t = \partial_t - \delta(t), \tag{2.5.1}$$

where $\delta(t)$, $t \in \mathbb{R}$, are symmetric derivations of the C^*-algebra A, i.e.,

$$\delta_t(ab) = \delta_t(a)b + a\delta_t(b), \qquad \delta_t(a^*) = \delta_t(a)^*, \qquad a, b \in A.$$

We say that a section α of the bundle of C^*-algebras $\mathcal{C} \to \mathbb{R}$ is an integral section of the connection ∇_t if

$$\nabla_t\alpha(t) = [\partial_t - \delta(t)]\alpha(t) = 0. \tag{2.5.2}$$

One can think of the equation (2.5.2) as being the *Heisenberg equation* describing quantum evolution.

In particular, let the derivations $\delta(t) = \delta$ in the Heisenberg equation (2.5.2) be the same for all $t \in \mathbb{R}$, and let δ be an infinitesimal generator of a strongly continuous one-parameter group $[G_t]$ of automorphisms of the C^*-algebra A [41]. The pair $(A, [G_t])$ is called the *C^*-dynamic system*. It describes evolution of a conservative quantum system. Namely, for any $a \in A$, the curve $\alpha(t) = G_t(a)$, $t \in \mathbb{R}$, in A is a unique solution with the initial value $\alpha(0) = a$ of the Heisenberg equation (2.5.2).

It should be emphasized that, if the derivation δ is unbounded, the connection ∇_t (2.5.1) is not defined everywhere on the algebra $\mathcal{C}(\mathbb{R})$. In this case, we deal with a generalized connection. It is given by operators of a parallel displacement, whose generators however are ill defined [4]. Moreover, it may happen that a representation π of the C^*-algebra A does not carry out a representation of the automorphism group $[G_t]$. Therefore,

quantum evolution described by the conservative Heisenberg equation, whose solution is a strongly (but not uniformly) continuous dynamic system (A, G_t), need not be described by the Schrödinger equation (see Remark 2.5.1 below).

If δ is a bounded derivation of a C^*-algebra A, the Heisenberg and Schrödinger pictures of evolution of a conservative quantum system are equivalent. Namely, as was mentioned above, δ is an infinitesimal generator of a uniformly continuous one-parameter group $[G_t]$ of automorphisms of A, and *vice versa*. For any representation π of A in a Hilbert space E, there exists a bounded self-adjoint operator $\mathcal{H}$ in E such that

$$\pi(\delta(a)) = -i[\mathcal{H}, \pi(a)], \qquad \pi(G_t) = \exp(-it\mathcal{H}), \quad a \in A, \qquad t \in \mathbb{R}. \tag{2.5.3}$$

The corresponding conservative Schrödinger equation reads

$$(\partial_t + i\mathcal{H})\psi = 0, \tag{2.5.4}$$

where ψ is a section of the trivial Hilbert bundle $\mathbb{R} \times E \to \mathbb{R}$. Its solution with an initial value $\psi(0) \in E$ is

$$\psi(t) = \exp[-it\mathcal{H}]\psi(0). \tag{2.5.5}$$

Remark 2.5.1: If the derivation δ is unbounded, but obeys some conditions, we also obtain the unitary representation (2.5.3) of the group $[G_t]$, but the curve $\psi(t)$ (2.5.5) need not be differentiable, and the Schrödinger equation (2.5.4) is ill defined [41]. $\diamondsuit$

Let us return to the general case of a non-conservative quantum system characterized by a bundle of C^*-algebras $\mathcal{C} \to \mathbb{R}$ with the typical fibre A. Let us suppose that a phase Hilbert space of a quantum system is preserved under evolution, i.e., instantaneous C^*-algebras A_t are endowed with representations equivalent to some representation of the C^*-algebra A in a Hilbert space E. Then quantum evolution can be described by means of the Schrödinger equation as follows.

Let us consider a smooth Hilbert bundle $\mathcal{E} \to \mathbb{R}$ with the typical fibre E and a connection ∇ on the $C^\infty(\mathbb{R})$-module $\mathcal{E}(\mathbb{R})$ of smooth sections of $\mathcal{E} \to \mathbb{R}$ (see previous Section). This connection assigns to the standard vector field ∂_t on $\mathbb{R}$ an $\mathbb{R}$-module endomorphism ∇_t of $\mathcal{E}(\mathbb{R})$ which obeys the Leibniz rule

$$\nabla_t(f\psi) = \partial_t f\psi + f\nabla_t\psi, \qquad \psi \in \mathcal{E}(\mathbb{R}), \qquad f \in C^\infty(\mathbb{R}),$$

and the condition

$$\langle(\nabla_t\psi)(t)|\psi(t)\rangle + \langle\psi(t)|(\nabla_t\psi)(t)\rangle = \partial_t\langle\psi(t)|\psi(t)\rangle.$$

Given a trivialization $\mathcal{E} = \mathbb{R} \times E$, the connection ∇_t reads

$$\nabla_t\psi = (\partial_t + i\mathcal{H}(t))\psi, \tag{2.5.6}$$

where $\mathcal{H}(t)$ are bounded self-adjoint operators in E for all $t \in \mathbb{R}$. It is a $U(E)$-principal connection.

We say that a section ψ of the Hilbert bundle $\Pi \to \mathbb{R}$ is an integral section of the connection ∇_t (2.5.6) if it fulfils the equation

$$\nabla_t \psi(t) = (\partial_t + i\mathcal{H}(t))\psi(t) = 0. \tag{2.5.7}$$

One can think of this equation as being the *Schrödinger equation* for the Hamiltonian $\mathcal{H}$. Its solution with an initial value $\psi(0) \in E$ exists and reads

$$\psi(t) = U(t)\psi(0), \tag{2.5.8}$$

where $U(t)$ is an *operator of a parallel displacement* with respect to the connection (2.5.6). This operator is a differentiable section of the trivial bundle

$$\mathbb{R} \times U(E) \to \mathbb{R}$$

which obeys the equation

$$\partial_t U(t) = -i\mathcal{H}(t)U(t), \qquad U_0 = \mathbf{1}. \tag{2.5.9}$$

The operator $U(t)$ plays the role of an *evolution operator*. It is given by the *time-ordered exponential*

$$U(t) = T \exp\left[-i \int_0^t \mathcal{H}(t')dt'\right], \tag{2.5.10}$$

which uniformly converges in the operator norm [29].

It should be emphasized that the evolution operator $U(t)$ has been defined with respect to a given trivialization of a Hilbert bundle $\mathcal{E} \to \mathbb{R}$.

2.6 Berry connection

Let us consider a quantum system depending on a finite number of real classical parameters given by sections of a smooth parameter bundle $\Sigma \to \mathbb{R}$. For the sake of simplicity, we fix a trivialization $\Sigma = \mathbb{R} \times Z$, coordinated by (t, σ^m). Although it may happen that the parameter bundle $\Sigma \to \mathbb{R}$ has no preferable trivialization.

In previous Section, we have characterized the time as a classical parameter in quantum mechanics. This characteristic is extended to other classical parameters. In a general setting, one assigns a C^*-algebra A_σ to each point $\sigma \in \Sigma$ of the parameter bundle Σ, and treat A_σ as a quantum system under fixed values (t, σ^m) of the parameters. However, we will simplify repeatedly our consideration in order to single out a desired Berry's phase phenomenon.

Let us assume that all algebras A_σ are isomorphic to the algebra $B(E)$ of bounded operators in some Hilbert space E, and consider a smooth Hilbert bundle $\mathcal{E} \to \Sigma$ with the typical fibre E. Smooth sections of $\mathcal{E} \to \Sigma$ constitute a module $\mathcal{E}(\Sigma)$ over the ring $C^\infty(\Sigma)$ of real functions on Σ. A connection $\widetilde{\nabla}$ on $\mathcal{E}(\Sigma)$ assigns to each vector field τ on Σ a first order differential operator

$$\widetilde{\nabla}_\tau \in \mathrm{Diff}\,_1(\mathcal{E}(\Sigma), \mathcal{E}(\Sigma)) \tag{2.6.1}$$

which obeys the Leibniz rule

$$\widetilde{\nabla}_\tau(fs) = (\tau\rfloor df)s + f\widetilde{\nabla}_\tau s, \qquad s \in \mathcal{E}(\Sigma), \qquad f \in C^\infty(\Sigma).$$

Let τ be a vector field on Σ such that $dt\rfloor\tau = 1$. Given a trivialization chart of the Hilbert bundle $\mathcal{E} \to \Sigma$, the operator $\widetilde{\nabla}_\tau$ (2.6.1) reads

$$\widetilde{\nabla}_\tau(s) = (\partial_t + i\mathcal{H}(t,\sigma^i))s + \tau^m(\partial_m - i\widehat{A}_m(t,\sigma^i))s, \tag{2.6.2}$$

where $\mathcal{H}(t,\sigma^i)$, $\widehat{A}_m(t,\sigma^i)$ for each $\sigma \in \Sigma$ are bounded self-adjoint operators in the Hilbert space E.

Let us consider the composite Hilbert bundle $\mathcal{E} \to \Sigma \to \mathbb{R}$. Every section $h(t)$ of the fibre bundle $\Sigma \to \mathbb{R}$ defines the subbundle $\mathcal{E}_h = h^*\mathcal{E} \to \mathbb{R}$ of the Hilbert bundle $\mathcal{E} \to \mathbb{R}$ whose typical fibre is E. Accordingly, the connection $\widetilde{\nabla}$ (2.6.2) on the $C^\infty(\Sigma)$-module $\mathcal{E}(\Sigma)$ yields the pull-back connection

$$\nabla_h(\psi) = [\partial_t - i(\widehat{A}_m(t,h^i(t))\partial_t h^m - \mathcal{H}(t,h^i(t))]\psi \tag{2.6.3}$$

on the $C^\infty(\mathbb{R})$-module $\mathcal{E}_h(\mathbb{R})$ of sections ψ of the fibre bundle $\mathcal{E}_h \to \mathbb{R}$.

We say that a section ψ of the fibre bundle $\mathcal{E}_h \to \mathbb{R}$ is an integral section of the connection (2.6.3) if

$$\nabla_h(\psi) = [\partial_t - i(\widehat{A}_m(t,h^i(t))\partial_t h^m - \mathcal{H}(t,h^i(t))]\psi = 0. \tag{2.6.4}$$

One can think of the equation (2.6.4) as being the Shrödinger equation for a quantum system depending on the parameter function $h(t)$. Its solutions take the form (2.5.8) where $U(t)$ is the time-ordered exponent

$$U(t) = T\exp\left[i\int_0^t(\widehat{A}_m\partial_{t'}h^m - \mathcal{H})dt'\right]. \tag{2.6.5}$$

The term $i\widehat{A}_m(t,h^i(t))\partial_t h^m$ in the Shrödinger equation (2.6.4) is responsible for the Berry's phase phenomenon, while $\mathcal{H}$ is treated as an ordinary Hamiltonian of a quantum system [41]. To show the Berry's phase phenomenon clearly, we will continue to simplify the system under consideration. Given a trivialization of the fibre bundle $\mathcal{E} \to \mathbb{R}$ and the above mentioned trivialization $\Sigma = \mathbb{R} \times Z$ of the parameter bundle Σ, let us suppose

that the components $\widehat{A}_m$ of the connection $\widetilde{\nabla}$ (2.6.2) are independent of t and that the operators $\mathcal{H}(\sigma)$ commute with the operators $\widehat{A}_m(\sigma')$ at all points of the curve $h(t) \subset \Sigma$. Then the operator $U(t)$ (2.6.5) takes the form

$$U(t) = T\exp\left[i \int\limits_{h([0,t])} \widehat{A}_m(\sigma^i)d\sigma^m\right] \cdot T\exp\left[-i\int\limits_0^t \mathcal{H}(t')dt'\right]. \tag{2.6.6}$$

One can think of the first factor in the right-hand side of the expression (2.6.6) as being the operator of a parallel transport along the curve $h([0,t]) \subset Z$ with respect to the pull-back connection

$$\nabla = i^*\widetilde{\nabla} = \partial_m - i\widehat{A}_m(t,\sigma^i) \tag{2.6.7}$$

on the fibre bundle $\mathcal{E} \to Z$, defined by the imbedding

$$i : Z \hookrightarrow \{0\} \times Z \subset \Sigma.$$

Note that, since $\widehat{A}_m$ are independent of time, one can utilize any imbedding of Z to $\{t\} \times Z$. Moreover, the connection ∇ (2.6.7), called the *Berry connection*, can be seen as a connection on some principal fibre bundle $P \to Z$ for the unitary group $U(E)$. Let the curve $h([0,t])$ be closed, while the holonomy group of the connection ∇ at the point $h(t) = h(0)$ is not trivial. Then the unitary operator

$$T\exp\left[i \int\limits_{h([0,t])} \widehat{A}_m(\sigma^i)d\sigma^m\right] \tag{2.6.8}$$

is not the identity. For example, if

$$i\widehat{A}_m(\sigma^i) = iA_m(\sigma^i)\mathrm{Id}_E \tag{2.6.9}$$

is a $U(1)$-principal connection on Z, then the operator (2.6.8) is the well-known Berry phase factor

$$\exp\left[i \int\limits_{h([0,t])} A_m(\sigma^i)d\sigma^m\right].$$

If (2.6.9) is a curvature-free connection, Berry's phase is exactly the Aharonov–Bohm effect on the parameter space Z.

The following variant of the Berry's phase phenomenon leads us to a principal bundle for familiar finite-dimensional Lie groups. Let a Hilbert space E be the Hilbert sum of n-dimensional eigenspaces of the Hamiltonian $\mathcal{H}(\sigma)$, i.e.,

$$E = \bigoplus_{k=1}^{\infty} E_k, \qquad E_k = P_k(E),$$

where P_k are the projection operators, i.e.,

$$H(\sigma) \circ P_k = \lambda_k(\sigma) P_k$$

(in the spirit of the adiabatic hypothesis). Let the operators $\widehat{A}_m(z)$ be time-independent and preserve the eigenspaces E_k of the Hamiltonian $\mathcal{H}$, i.e.,

$$\widehat{A}_m(z) = \sum_k \widehat{A}^k_m(z) P_k, \tag{2.6.10}$$

where $\widehat{A}^k_m(z)$, $z \in Z$, are self-adjoint operators in E_k. It follows that $\widehat{A}_m(\sigma)$ commute with $\mathcal{H}(\sigma)$ at all points of the parameter bundle $\Sigma \to \mathbb{R}$. Then, restricted to each subspace E_k, the parallel transport operator (2.6.8) is a unitary operator in E_k. In this case, the Berry connection (2.6.7) on the $U(E)$-principal bundle $P \to Z$ can be seen as a composite connection on the composite bundle

$$P \to P/U(n) \to Z,$$

which is defined by some principal connection on the $U(n)$-principal bundle $P \to P/U(n)$ and the trivial connection on the fibre bundle $P/U(n) \to Z$ [41].

Note that, since $U(E)$ is contractible, the $U(n)$-principal bundle $U(E) \to U(E)/U(n)$ is universal and, consequently, the typical fibre $U(E)/U(n)$ of $P/U(n) \to Z$ is exactly the classifying space $B(U(n)$ of $U(n)$-principal bundles [61]. Moreover, one can consider the parallel transport along a curve in the bundle $P/U(n)$. In this case, a state vector $\psi(t)$ acquires a geometric phase factor in addition to the dynamical phase factor. In particular, if $\Sigma = \mathbb{R}$ (i.e., classical parameters are absent and Berry's phase has only the geometric origin) we come to the case of a Berry connection on the $U(n)$-principal bundle over the classifying space $B(U(n))$ [8]. If $n = 1$, this is the variant of Berry's geometric phase of [2].

Chapter 3

Supergeometry

Supergeometry is phrased in terms of $\mathbb{Z}_2$-graded modules and sheaves over $\mathbb{Z}_2$-graded commutative algebras. Their algebraic properties naturally generalize those of modules and sheaves over commutative algebras, but this is not a particular case of non-commutative geometry because of the peculiar definition of graded derivations.

3.1 Graded tensor calculus

Unless otherwise stated, by a graded structure throughout this Chapter is meant a $\mathbb{Z}_2$-graded structure, and the symbol [.] stands for the $\mathbb{Z}_2$-graded parity. Let us recall some basic notions of the graded tensor calculus [5, 20, 80].

An algebra $\mathcal{A}$ is called *graded* if it is endowed with a *grading automorphism* γ such that $\gamma^2 = \mathrm{Id}$. A graded algebra seen as a $\mathbb{Z}$-module falls into the direct sum $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ of two $\mathbb{Z}$-modules $\mathcal{A}_0$ and $\mathcal{A}_1$ of *even* and *odd* elements such that

$$\gamma(a) = (-1)^i a, \qquad a \in \mathcal{A}_i, \qquad i = 0, 1.$$

One calls $\mathcal{A}_0$ and $\mathcal{A}_1$ the even and odd parts of $\mathcal{A}$, respectively. In particular, if $\gamma = \mathrm{Id}$, then $\mathcal{A} = \mathcal{A}_0$. Since

$$\gamma(aa') = \gamma(a)\gamma(a'),$$

we have

$$[aa'] = ([a] + [a']) \bmod 2$$

where $a \in \mathcal{A}_{[a]}$, $a' \in \mathcal{A}_{[a']}$. It follows that $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$ and $\mathcal{A}_1$ is an $\mathcal{A}_0$-module. If $\mathcal{A}$ is a graded ring, then $[\mathbf{1}] = 0$.

A graded algebra $\mathcal{A}$ is said to be *graded commutative* if

$$aa' = (-1)^{[a][a']} a'a,$$

where a and a' are arbitrary *homogeneous elements* of $\mathcal{A}$, i.e., they are either even or odd.

Given a graded algebra $\mathcal{A}$, a left *graded $\mathcal{A}$-module* Q is a left $\mathcal{A}$-module provided with the grading automorphism γ such that

$$\gamma(aq) = \gamma(a)\gamma(q), \qquad a \in \mathcal{A}, \qquad q \in Q,$$

i.e.,

$$[aq] = ([a] + [q]) \bmod 2.$$

A graded module Q is split into the direct sum $Q = Q_0 \oplus Q_1$ of two $\mathcal{A}_0$-modules Q_0 and Q_1 of even and odd elements. Similarly, right graded modules are defined.

If $\mathcal{K}$ is a graded commutative ring, a graded $\mathcal{K}$-module can be provided with a graded *$\mathcal{K}$-bimodule* structure by letting

$$qa = (-1)^{[a][q]} aq, \qquad a \in \mathcal{K}, \qquad q \in Q.$$

A graded $\mathcal{K}$-module is called *free* if it has a basis generated by homogeneous elements. This basis is said to be of type (n, m) if it contains n even and m odd elements.

In particular, by a (real) *graded vector space* $B = B_0 \oplus B_1$ is meant a graded $\mathbb{R}$-module. A graded vector space is said to be (n, m)-dimensional if $B_0 = \mathbb{R}^n$ and $B_1 = \mathbb{R}^m$.

The following are standard constructions of new graded modules from old ones.

• The direct sum of graded modules over the same graded commutative ring and a graded factor module are defined just as those of modules over a commutative ring.

• The *tensor product* $P \otimes Q$ of graded $\mathcal{K}$-modules P and Q is an additive group generated by elements $p \otimes q$, $p \in P$, $q \in Q$, obeying the relations

$$\begin{aligned}
&(p + p') \otimes q = p \otimes q + p' \otimes q, \\
&p \otimes (q + q') = p \otimes q + p \otimes q', \\
&ap \otimes q = (-1)^{[p][a]} pa \otimes q = (-1)^{[p][a]} p \otimes aq = \\
&\qquad (-1)^{([p]+[q])[a]} p \otimes qa, \qquad a \in \mathcal{K}.
\end{aligned}$$

In particular, the tensor algebra $\otimes Q$ of a graded $\mathcal{K}$-module Q is defined as that of a module over a commutative ring in Example 1.1.2. Its quotient $\wedge Q$ with respect to the ideal generated by elements

$$q \otimes q' + (-1)^{[q][q']} q' \otimes q, \qquad q, q' \in Q,$$

is the *bigraded exterior algebra* of a graded module Q with respect to the *graded exterior product*

$$q \wedge q' = -(-1)^{[q][q']} q' \wedge q.$$

• A morphism $\Phi : P \to Q$ of graded $\mathcal{K}$-modules is said to be *even* (resp. *odd*) if Φ preserves (resp. change) the graded parity of all elements P. It obeys the relations

$$\Phi(ap) = (-1)^{[\Phi][a]}\Phi(p), \qquad p \in P, \qquad a \in \mathcal{K}.$$

The set $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ of graded morphisms of a graded $\mathcal{K}$-module P to a graded $\mathcal{K}$-module Q is naturally a graded $\mathcal{K}$-module. The graded $\mathcal{K}$-module $P^* = \mathrm{Hom}_{\mathcal{K}}(P, \mathcal{K})$ is called the *dual* of a graded $\mathcal{K}$-module P.

A *graded commutative $\mathcal{K}$-ring* A is a graded commutative ring which is also a graded $\mathcal{K}$-module. A graded commutative $\mathbb{R}$-ring is said to be of rank N if it is a free algebra generated by the unit $\mathbf{1}$ and N odd elements. A *graded commutative Banach ring* A is a graded commutative $\mathbb{R}$-ring which is a real Banach algebra whose norm obeys the additional condition

$$\|a_0 + a_1\| = \|a_0\| + \|a_1\|, \qquad a_0 \in A_0, \quad a_1 \in A_1.$$

Let V be a real vector space. Let $\Lambda = \wedge V$ be its ($\mathbb{N}$-graded) exterior algebra provided with the $\mathbb{Z}_2$-graded structure

$$\Lambda = \Lambda_0 \oplus \Lambda_1, \qquad \Lambda_0 = \mathbb{R} \bigoplus_{k=1} \overset{2k}{\wedge} V, \qquad \Lambda_1 = \bigoplus_{k=1} \overset{2k-1}{\wedge} V. \tag{3.1.1}$$

It is a graded commutative $\mathbb{R}$-ring, called the *Grassmann algebra*. A Grassmann algebra, seen as an additive group, admits the decomposition

$$\Lambda = \mathbb{R} \oplus R = \mathbb{R} \oplus R_0 \oplus R_1 = \mathbb{R} \oplus (\Lambda_1)^2 \oplus \Lambda_1, \tag{3.1.2}$$

where R is the *ideal of nilpotents* of Λ. The corresponding projections $\sigma : \Lambda \to \mathbb{R}$ and $s : \Lambda \to R$ are called the *body* and *soul* maps, respectively.

Hereafter, we restrict our consideration to Grassmann algebras of finite rank. Given a basis $\{c^i\}$ for the vector space V, the elements of the Grassmann algebra Λ (3.1.1) take the form

$$a = \sum_{k=0} \sum_{(i_1 \cdots i_k)} a_{i_1 \cdots i_k} c^{i_1} \cdots c^{i_k}, \tag{3.1.3}$$

where the second sum runs through all the tuples $(i_1 \cdots i_k)$ such that no two of them are permutations of each other. The Grassmann algebra Λ becomes a graded commutative Banach ring if its elements (3.1.3) are endowed with the norm

$$\|a\| = \sum_{k=0} \sum_{(i_1 \cdots i_k)} |a_{i_1 \cdots i_k}|.$$

Let B be a graded vector space. Given a Grassmann algebra Λ of rank N, it can be brought into a graded Λ-module

$$\Lambda B = (\Lambda B)_0 \oplus (\Lambda B)_1 = (\Lambda_0 \otimes B_0 \oplus \Lambda_1 \otimes B_1) \oplus (\Lambda_1 \otimes B_0 \oplus \Lambda_0 \otimes B_1),$$

called a *superspace*. The superspace

$$B^{n|m} = [(\overset{n}{\oplus}\Lambda_0) \oplus (\overset{m}{\oplus}\Lambda_1)] \oplus [(\overset{n}{\oplus}\Lambda_1) \oplus (\overset{m}{\oplus}\Lambda_0)] \tag{3.1.4}$$

is said to be (n,m)-dimensional. The graded Λ_0-module

$$B^{n,m} = (\overset{n}{\oplus}\Lambda_0) \oplus (\overset{m}{\oplus}\Lambda_1)$$

is called an (n,m)-dimensional *supervector space*.

Whenever referring to a topology on a supervector space $B^{n,m}$, we will mean the Euclidean topology on a $2^{N-1}[n+m]$-dimensional real vector space.

Given a superspace $B^{n|m}$ over a Grassmann algebra Λ, a Λ-module endomorphism of $B^{n|m}$ is represented by an $(n+m)\times(n+m)$ matrix

$$L = \begin{pmatrix} L_1 \, L_2 \\ L_3 \, L_4 \end{pmatrix} \tag{3.1.5}$$

with entries in Λ. It is called a *supermatrix*. One says that a supermatrix L is

- *even* if L_1 and L_4 have even entries, while L_2 and L_3 have the odd ones;
- *odd* if L_1 and L_4 have odd entries, while L_2 and L_3 have the even ones.

Endowed with this gradation, the set of supermatrices (3.1.5) is a graded Λ-ring.

Invertible supermatrices constitute a group $GL(n|m;\Lambda)$, called the *general linear graded group*.

Let $\mathcal{K}$ be a graded commutative ring. A graded commutative (non-associative) $\mathcal{K}$-algebra $\mathfrak{g}$ is called a *Lie $\mathcal{K}$-superalgebra* if its product, called the *superbracket* and denoted by $[.,.]$, obeys the relations

$$[\varepsilon,\varepsilon'] = -(-1)^{[\varepsilon][\varepsilon']}[\varepsilon',\varepsilon],$$
$$(-1)^{[\varepsilon][\varepsilon'']}[\varepsilon,[\varepsilon',\varepsilon'']] + (-1)^{[\varepsilon'][\varepsilon]}[\varepsilon',[\varepsilon'',\varepsilon]] + (-1)^{[\varepsilon''][\varepsilon']}[\varepsilon'',[\varepsilon,\varepsilon']] = 0.$$

Obviously, the even part $\mathfrak{g}_0$ of a Lie $\mathcal{K}$-superalgebra $\mathfrak{g}$ is a Lie $\mathcal{K}_0$-algebra. A graded $\mathcal{K}$-module P is called a $\mathfrak{g}$-module if it is provided with a $\mathcal{K}$-bilinear map

$$\mathfrak{g}\times P \ni (\varepsilon,p) \mapsto \varepsilon p \in P,$$
$$[\varepsilon p] = ([\varepsilon]+[p]) \bmod 2,$$
$$[\varepsilon,\varepsilon']p = (\varepsilon\circ\varepsilon' - (-1)^{[\varepsilon][\varepsilon']}\varepsilon'\circ\varepsilon)p.$$

3.2 Graded differential calculus and connections

Linear differential operators and connections on graded modules over graded commutative rings are defined similarly to those in commutative geometry [41, 80].

Let $\mathcal{K}$ be a graded commutative ring and $\mathcal{A}$ a graded commutative $\mathcal{K}$-ring. Let P and Q be graded $\mathcal{A}$-modules. The graded $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ of graded $\mathcal{K}$-module homomorphisms $\Phi : P \to Q$ can be endowed with the two graded $\mathcal{A}$-module structures

$$(a\Phi)(p) = a\Phi(p), \qquad (\Phi \bullet a)(p) = \Phi(ap), \qquad a \in \mathcal{A}, \quad p \in P, \tag{3.2.1}$$

called $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures, respectively. Let us put

$$\delta_a \Phi = a\Phi - (-1)^{[a][\Phi]}\Phi \bullet a, \qquad a \in \mathcal{A}. \tag{3.2.2}$$

An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is said to be a Q-valued *graded differential operator* of order s on P if

$$\delta_{a_0} \circ \cdots \circ \delta_{a_s} \Delta = 0$$

for any tuple of $s+1$ elements $a_0, \ldots, a_s$ of $\mathcal{A}$. The set $\mathrm{Diff}_s(P,Q)$ of these operators inherits the graded module structures (3.2.1).

In particular, zero order graded differential operators obey the condition

$$\delta_a \Delta(p) = a\Delta(p) - (-1)^{[a][\Delta]}\Delta(ap) = 0, \qquad a \in \mathcal{A}, \qquad p \in P,$$

i.e., they coincide with graded $\mathcal{A}$-module morphisms $P \to Q$. A first order graded differential operator Δ satisfies the condition

$$\begin{aligned}\delta_a \circ \delta_b \, \Delta(p) = ab\Delta(p) - (-1)^{([b]+[\Delta])[a]} b\Delta(ap) - (-1)^{[b][\Delta]} a\Delta(bp) + \\ (-1)^{[b][\Delta]+([\Delta]+[b])[a]} = 0, \qquad a, b \in \mathcal{A}, \quad p \in P.\end{aligned}$$

For instance, let $P = \mathcal{A}$. Any zero order Q-valued graded differential operator Δ on $\mathcal{A}$ is defined by its value $\Delta(\mathbf{1})$. Then there is a graded $\mathcal{A}$-module isomorphism

$$\mathrm{Diff}_0(\mathcal{A}, Q) = Q$$

via the association

$$Q \ni q \mapsto \Delta_q \in \mathrm{Diff}_0(\mathcal{A}, Q),$$

where Δ_q is given by the equality $\Delta_q(\mathbf{1}) = q$. A first order Q-valued graded differential operator Δ on $\mathcal{A}$ fulfils the condition

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]}a\Delta(b) - (-1)^{([b]+[a])[\Delta]}ab\Delta(\mathbf{1}), \qquad a, b \in \mathcal{A}.$$

It is called a Q-valued *graded derivation* of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$, i.e., the graded Leibniz rule

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]}a\Delta(b), \qquad a, b \in \mathcal{A}, \tag{3.2.3}$$

holds (cf. (1.2.10)). One obtains at once that any first order graded differential operator on $\mathcal{A}$ falls into the sum

$$\Delta(a) = \Delta(\mathbf{1})a + [\Delta(a) - \Delta(\mathbf{1})a]$$

of a zero order graded differential operator $\Delta(\mathbf{1})a$ and a graded derivation $\Delta(a) - \Delta(\mathbf{1})a$. If ∂ is a graded derivation of $\mathcal{A}$, then $a\partial$ is so for any $a \in \mathcal{A}$. Hence, graded derivations of $\mathcal{A}$ constitute a graded $\mathcal{A}$-module $\mathfrak{d}(\mathcal{A}, Q)$, called the *graded derivation module*.

If $Q = \mathcal{A}$, the graded derivation module $\mathfrak{d}\mathcal{A}$ is also a Lie superalgebra over the graded commutative ring $\mathcal{K}$ with respect to the superbracket

$$[u, u'] = u \circ u' - (-1)^{[u][u']} u' \circ u, \qquad u, u' \in \mathcal{A}. \tag{3.2.4}$$

We have the graded $\mathcal{A}$-module decomposition

$$\mathrm{Diff}_1(\mathcal{A}) = \mathcal{A} \oplus \mathfrak{d}\mathcal{A}. \tag{3.2.5}$$

Let us turn now to jets of graded modules. Given a graded $\mathcal{A}$-module P, let us consider the tensor product $\mathcal{A} \otimes_{\mathcal{K}} P$ of graded $\mathcal{K}$-modules $\mathcal{A}$ and P. We put

$$\delta^b(a \otimes p) = (ba) \otimes p - (-1)^{[a][b]} a \otimes (bp), \qquad p \in P, \quad a, b \in \mathcal{A}. \tag{3.2.6}$$

The *k-order graded jet module* $\mathcal{J}^k(P)$ of the module P is defined as the quotient of the graded $\mathcal{K}$-module $\mathcal{A} \otimes_{\mathcal{K}} P$ by its submodule generated by elements of type

$$\delta^{b_0} \circ \cdots \circ \delta^{b_k}(a \otimes p).$$

In particular, the first order graded jet module $\mathcal{J}^1(P)$ consists of elements $a \otimes_1 p$ modulo the relations

$$ab \otimes_1 p - (-1)^{[a][b]} b \otimes_1 (ap) - a \otimes_1 (bp) + \mathbf{1} \otimes_1 (abp) = 0. \tag{3.2.7}$$

For any $h \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes P, Q)$, the equality

$$\delta_b(h(a \otimes p)) = (-1)^{[h][b]} h(\delta^b(a \otimes p))$$

holds. By analogous with Theorem 1.2.3, one then can show that any Q-valued graded differential operator Δ of order k on a graded $\mathcal{A}$-module P factorizes uniquely

$$\Delta : P \xrightarrow{J^k} \mathcal{J}^k(P) \longrightarrow Q$$

through the morphism

$$J^k : p \ni p \mapsto \mathbf{1} \otimes_k p \in \mathcal{J}^k(P)$$

and some homomorphism $\mathfrak{f}^\Delta : \mathcal{J}^k(P) \to Q$. Accordingly, the assignment $\Delta \mapsto \mathfrak{f}^\Delta$ defines an isomorphism

$$\mathrm{Diff}_s(P, Q) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^s(P), Q). \tag{3.2.8}$$

Let us focus on the first order graded jet module $\mathcal{J}^1$ of $\mathcal{A}$ consisting of the elements $a \otimes_1 b$, $a, b \in \mathcal{A}$, subject to the relations

$$ab \otimes_1 \mathbf{1} - (-1)^{[a][b]} b \otimes_1 a - a \otimes_1 b + \mathbf{1} \otimes_1 (ab) = 0. \tag{3.2.9}$$

It is endowed with the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures

$$c(a \otimes_1 b) = (ca) \otimes_1 b, \qquad c \bullet (a \otimes_1 b) = a \otimes_1 (cb).$$

There are canonical $\mathcal{A}$- and $\mathcal{A}^\bullet$-module monomorphisms

$$\begin{aligned} &i_1 : \mathcal{A} \ni a \mapsto a \otimes_1 \mathbf{1} \in \mathcal{J}^1, \\ &J^1 : \mathcal{A} \ni a \mapsto \mathbf{1} \otimes_1 a \in \mathcal{J}^1, \end{aligned}$$

such that $\mathcal{J}^1$, seen as a graded $\mathcal{A}$-module, is generated by the elements $J^1 a$, $a \in \mathcal{A}$. With these monomorphisms, we have the canonical $\mathcal{A}$-module splitting

$$\begin{aligned} &\mathcal{J}^1 = i_1(\mathcal{A}) \oplus \mathcal{O}^1, \\ &aJ^1(b) = a \otimes_1 b = ab \otimes_1 \mathbf{1} + a(\mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1}), \end{aligned} \tag{3.2.10}$$

where the graded $\mathcal{A}$-module $\mathcal{O}^1$ is generated by the elements

$$\mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1}, \qquad b \in \mathcal{A}.$$

Let us consider the corresponding $\mathcal{A}$-module epimorphism

$$h^1 : \mathcal{J}^1 \ni \mathbf{1} \otimes_1 b \mapsto \mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1} \in \mathcal{O}^1 \tag{3.2.11}$$

and the composition

$$d = h^1 \circ J_1 : \mathcal{A} \ni b \mapsto \mathbf{1} \otimes_1 b - b \otimes_1 \mathbf{1} \in \mathcal{O}^1. \tag{3.2.12}$$

The equality

$$d(ab) = a \otimes_1 b + b \otimes_1 a - ab \otimes_1 \mathbf{1} - ba \otimes_1 \mathbf{1} = (-1)^{[a][b]} bda + adb$$

shows that d (3.2.12) is an even $\mathcal{O}^1$-valued derivation of $\mathcal{A}$. Seen as a graded $\mathcal{A}$-module, $\mathcal{O}^1$ is generated by the elements da for all $a \in \mathcal{A}$.

In view of the splittings (3.2.5) and (3.2.10), the isomorphism (3.2.8) reduces to the isomorphism

$$\mathfrak{d}\mathcal{A} = \mathcal{O}^{1*} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}^1, \mathcal{A}) \tag{3.2.13}$$

of $\mathfrak{d}\mathcal{A}$ to the dual $\mathcal{O}^{1*}$ of the graded $\mathcal{A}$-module $\mathcal{O}^1$. It is given by the duality relations

$$\mathfrak{d}\mathcal{A} \ni u \leftrightarrow \phi_u \in \mathcal{O}^{1*}, \qquad \phi_u(da) = u(a), \qquad a \in \mathcal{A}. \tag{3.2.14}$$

Using this fact, let us construct a differential calculus over a graded commutative $\mathcal{K}$-ring $\mathcal{A}$.

Let us consider the bigraded exterior algebra $\mathcal{O}^*$ of a graded module $\mathcal{O}^1$. It consists of finite linear combinations of monomials of the form

$$\phi = a_0 da_1 \wedge \cdots \wedge da_k, \qquad a_i \in \mathcal{A}, \tag{3.2.15}$$

whose product obeys the juxtaposition rule

$$(a_0da_1)\wedge(b_0db_1) = a_0d(a_1b_0)\wedge db_1 - a_0a_1db_0\wedge db_1$$

and the bigraded commutative relations

$$\phi\wedge\phi' = (-1)^{|\phi||\phi'|+[\phi][\phi']}\phi'\wedge\phi. \tag{3.2.16}$$

In order to make $\mathcal{O}^*$ to a differential algebra, let us define the coboundary operator $d:\mathcal{O}^1\to\mathcal{O}^2$ by the rule

$$d\phi(u,u') = -u'(u(\phi)) + (-1)^{[u][u']}u(u'(\phi)) + [u',u](\phi),$$

where $u,u'\in\mathfrak{d}\mathcal{A}$, $\phi\in\mathcal{O}^1$, and u, u' are both graded derivatives of $\mathcal{A}$ and $\mathcal{A}$-valued forms on $\mathcal{O}^1$. It is readily observed that, by virtue of the relation (3.2.14), $(d\circ d)(a)=0$ for all $a\in\mathcal{A}$. Then d is extended to the bigraded exterior algebra $\mathcal{O}^*$ if its action on monomials (3.2.15) is defined as

$$d(a_0da_1\wedge\cdots\wedge da_k) = da_0\wedge da_1\wedge\cdots\wedge da_k.$$

This operator is nilpotent and fulfils the familiar relations

$$d(\phi\wedge\phi') = d\phi\wedge\phi' + (-1)^{|\phi||\phi'|}\phi\wedge d\phi'. \tag{3.2.17}$$

It makes $\mathcal{O}^*$ into a *differential bigraded algebra*, called a *graded differential calculus* over a graded commutative $\mathcal{K}$-ring $\mathcal{A}$.

Furthermore, one can extend the duality relation (3.2.14) to the *graded interior product* of $u\in\mathfrak{d}\mathcal{A}$ with any monomial ϕ (3.2.15) by the rules

$$\begin{aligned}&u\rfloor(bda) = (-1)^{[u][b]}u(a),\\ &u\rfloor(\phi\wedge\phi') = (u\rfloor\phi)\wedge\phi' + (-1)^{|\phi|+[\phi][u]}\phi\wedge(u\rfloor\phi').\end{aligned} \tag{3.2.18}$$

As a consequence, any graded derivation $u\in\mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ yields a derivation

$$\begin{aligned}&\mathbf{L}_u\phi = u\rfloor d\phi + d(u\rfloor\phi), \qquad \phi\in\mathcal{O}^*, \qquad u\in\mathfrak{d}\mathcal{A},\\ &\mathbf{L}_u(\phi\wedge\phi') = \mathbf{L}_u(\phi)\wedge\phi' + (-1)^{[u][\phi]}\phi\wedge\mathbf{L}_u(\phi'),\end{aligned} \tag{3.2.19}$$

of the bigraded algebra $\mathcal{O}^*$ called the *graded Lie derivative* of $\mathcal{O}^*$.

Remark 3.2.1: Since $\mathfrak{d}\mathcal{A}$ is a Lie $\mathcal{K}$-superalgebra, let us consider the Chevalley–Eilenberg complex $C^*[\mathfrak{d}\mathcal{A};\mathcal{A}]$ where the graded commutative ring $\mathcal{A}$ is a regarded as a $\mathfrak{d}\mathcal{A}$-module [39]. It is the complex

$$0\to\mathcal{A}\xrightarrow{\delta^0}C^1[\mathfrak{d}\mathcal{A};\mathcal{A}]\xrightarrow{\delta^1}\cdots C^k[\mathfrak{d}\mathcal{A};\mathcal{A}]\xrightarrow{\delta^k}\cdots \tag{3.2.20}$$

where

$$C^k[\mathfrak{d}\mathcal{A};\mathcal{A}] = \mathrm{Hom}_{\mathcal{K}}(\overset{k}{\wedge}\mathfrak{d}\mathcal{A},\mathcal{A})$$

are $\mathfrak{d}\mathcal{A}$-modules of $\mathcal{K}$-linear graded morphisms of the graded exterior products $\overset{k}{\wedge}\mathfrak{d}\mathcal{A}$ of the $\mathcal{K}$-module $\mathfrak{d}\mathcal{A}$ to $\mathcal{A}$. Let us bring homogeneous elements of $\overset{k}{\wedge}\mathfrak{d}\mathcal{A}$ into the form

$$\varepsilon_1 \wedge \cdots \varepsilon_r \wedge \epsilon_{r+1} \wedge \cdots \wedge \epsilon_k, \qquad \varepsilon_i \in \mathfrak{d}\mathcal{A}_0, \quad \epsilon_j \in \mathfrak{d}\mathcal{A}_1.$$

Then the coboundary operators of the complex (3.2.20) are given by the expression

$$\begin{aligned}
&\delta^{r+s-1}c(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \wedge \epsilon_s) = \\
&\sum_{i=1}^{r}(-1)^{i-1}\varepsilon_i c(\varepsilon_1 \wedge \cdots \widehat{\varepsilon}_i \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \epsilon_s) + \\
&\sum_{j=1}^{s}(-1)^{r}\varepsilon_i c(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \widehat{\epsilon}_j \cdots \wedge \epsilon_s) + \\
&\sum_{1\leq i<j\leq r}(-1)^{i+j}c([\varepsilon_i,\varepsilon_j] \wedge \varepsilon_1 \wedge \cdots \widehat{\varepsilon}_i \cdots \widehat{\varepsilon}_j \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \wedge \epsilon_s) + \\
&\sum_{1\leq i<j\leq s} c([\epsilon_i,\epsilon_j] \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \widehat{\epsilon}_i \cdots \widehat{\epsilon}_j \cdots \wedge \epsilon_s) + \\
&\sum_{1\leq i<r,1\leq j\leq s}(-1)^{i+r+1}c([\varepsilon_i,\epsilon_j] \wedge \varepsilon_1 \wedge \cdots \widehat{\varepsilon}_i \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \widehat{\epsilon}_j \cdots \wedge \epsilon_s).
\end{aligned} \tag{3.2.21}$$

The subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of the complex (3.2.20) of $\mathcal{A}$-linear morphisms is the graded Chevalley–Eilenberg differential calculus over a graded commutative $\mathcal{K}$-ring $\mathcal{A}$. Then one can show that the above mentioned graded differential calculus $\mathcal{O}^*$ is a subcomplex of the Chevalley–Eilenberg one $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$. $\diamondsuit$

Following the construction of a connection in commutative geometry in Section 1.3, one comes to the notion of a connection on modules over a graded commutative $\mathbb{R}$-ring $\mathcal{A}$. The following is a straightforward counterpart of Definitions 1.3.3 and 1.3.4.

DEFINITION 3.2.1: A *connection* on a graded $\mathcal{A}$-module P is an $\mathcal{A}$-module morphism

$$\mathfrak{d}\mathcal{A} \ni u \mapsto \nabla_u \in \mathrm{Diff}_1(P,P) \tag{3.2.22}$$

such that the first order differential operators ∇_u obey the Leibniz rule

$$\nabla_u(ap) = u(a)p + (-1)^{[a][u]}a\nabla_u(p), \quad a \in \mathcal{A}, \quad p \in P. \tag{3.2.23}$$

$\Box$

DEFINITION 3.2.2: Let P in Definition 3.2.1 be a graded commutative $\mathcal{A}$-ring and $\mathfrak{d}P$ the derivation module of P as a graded commutative $\mathcal{K}$-ring. A *connection* on a graded commutative $\mathcal{A}$-ring P is a $\mathcal{A}$-module morphism

$$\mathfrak{d}\mathcal{A} \ni u \mapsto \nabla_u \in \mathfrak{d}P, \tag{3.2.24}$$

which is a connection on P as an $\mathcal{A}$-module, i.e., obeys the Leibniz rule (3.2.23). $\Box$

3.3 Geometry of graded manifolds

By a *graded manifold* of dimension (n,m) is meant a local-ringed space $(Z,\mathfrak{A})$ where Z is an n-dimensional smooth manifold Z and $\mathfrak{A}=\mathfrak{A}_0\oplus\mathfrak{A}_1$ is a sheaf of graded commutative algebras of rank m such that [5]:

(i) there is the exact sequence of sheaves

$$0\to\mathcal{R}\to\mathfrak{A}\xrightarrow{\sigma}C^\infty_Z\to 0,\qquad \mathcal{R}=\mathfrak{A}_1+(\mathfrak{A}_1)^2, \tag{3.3.1}$$

where C^∞_Z is the sheaf of smooth real functions on Z;

(ii) $\mathcal{R}/\mathcal{R}^2$ is a locally free sheaf of C^∞_Z-modules of finite rank (with respect to pointwise operations), and the sheaf $\mathfrak{A}$ is locally isomorphic to the the exterior product $\wedge_{C^\infty_Z}(\mathcal{R}/\mathcal{R}^2)$.

The sheaf $\mathfrak{A}$ is called a *structure sheaf* of the graded manifold $(Z,\mathfrak{A})$, while the manifold Z is said to be a *body* of $(Z,\mathfrak{A})$. Sections of the sheaf $\mathfrak{A}$ are called *graded functions*. They make up a graded commutative $C^\infty(Z)$-ring $\mathfrak{A}(Z)$.

A graded manifold $(Z,\mathfrak{A})$ has the following local structure. Given a point $z\in Z$, there exists its open neighborhood U, called a *splitting domain*, such that

$$\mathfrak{A}(U)\cong C^\infty(U)\otimes\wedge\mathbb{R}^m. \tag{3.3.2}$$

It means that the restriction $\mathfrak{A}|_U$ of the structure sheaf $\mathfrak{A}$ to U is isomorphic to the sheaf $C^\infty_U\otimes\wedge\mathbb{R}^m$ of sections of some exterior bundle

$$\wedge E^*_U=U\times\wedge\mathbb{R}^m\to U.$$

The well-known *Batchelor's theorem* [5, 6, 41] states that such a structure of graded manifolds is global.

THEOREM 3.3.1: Let $(Z,\mathfrak{A})$ be a graded manifold. There exists a vector bundle $E\to Z$ with an m-dimensional typical fibre V such that the structure sheaf $\mathfrak{A}$ of $(Z,\mathfrak{A})$ is isomorphic to the structure sheaf $\mathfrak{A}_E$ of sections of the exterior bundle $\wedge E^*$, whose typical fibre is the Grassmann algebra $\wedge V^*$. □

It should be emphasized that Batchelor's isomorphism in Theorem 3.3.1 fails to be canonical. At the same time, there are many physical models where a vector bundle E is introduced from the beginning. In this case, it suffices to consider the structure sheaf $\mathfrak{A}_E$ of the exterior bundle $\wedge E^*$. We agree to call the pair $(Z,\mathfrak{A}_E)$ a *simple graded manifold*. Its automorphisms are restricted to those induced by automorphisms of the vector bundle $E\to Z$, called the *characteristic vector bundle* of the simple graded manifold $(Z,\mathfrak{A}_E)$. Accordingly, the structure module

$$\mathfrak{A}_E(Z)=\wedge E^*(Z)$$

of the sheaf $\mathfrak{A}_E$ (and of the exterior bundle $\wedge E^*$) is said to be the *structure module of the simple graded manifold* $(Z,\mathfrak{A}_E)$.

Given a simple graded manifold $(Z, \mathfrak{A}_E)$, every trivialization chart $(U; z^A, y^a)$ of the vector bundle $E \to Z$ is a splitting domain of $(Z, \mathfrak{A}_E)$. Graded functions on such a chart are Λ-valued functions

$$f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1 \ldots a_k}(z) c^{a_1} \cdots c^{a_k}, \tag{3.3.3}$$

where $f_{a_1 \cdots a_k}(z)$ are smooth functions on U and $\{c^a\}$ is the fibre basis for E^*. In particular, the sheaf epimorphism σ in (3.3.1) is induced by the body map of Λ. We agree to call $\{z^A, c^a\}$ the local *basis* for the graded manifold $(Z, \mathfrak{A}_E)$. Transition functions

$$y'^a = \rho^a_b(z^A) y^b$$

of bundle coordinates on $E \to Z$ induce the corresponding transformation

$$c'^a = \rho^a_b(z^A) c^b \tag{3.3.4}$$

of the associated local basis for the graded manifold $(Z, \mathfrak{A}_E)$ and the according coordinate transformation law of graded functions (3.3.3).

Remark 3.3.1: Although graded functions are locally represented by Λ-valued functions (3.3.3), they are not Λ-valued functions on a manifold Z because of the transformation law (3.3.4). $\diamondsuit$

Remark 3.3.2: Let us note that general automorphisms of a graded manifold take the form

$$c'^a = \rho^a(z^A, c^b), \tag{3.3.5}$$

where $\rho^a(z^A, c^b)$ are local graded functions. Considering simple graded manifolds, we actually restrict the class of graded manifold transformations (3.3.5) to the linear ones (3.3.4), compatible with given Batchelor's isomorphism. $\diamondsuit$

Let $E \to Z$ and $E' \to Z$ be vector bundles and $\Phi : E \to E'$ their bundle morphism over a morphism $\zeta : Z \to Z'$. Then every section s^* of the dual bundle $E'^* \to Z'$ defines the pull-back section $\Phi^* s^*$ of the dual bundle $E^* \to Z$ by the law

$$v_z \rfloor \Phi^* s^*(z) = \Phi(v_z) \rfloor s^*(\zeta(z)), \qquad v_z \in E_z.$$

It follows that a linear bundle morphism Φ yields a morphism

$$S\Phi : (Z, \mathfrak{A}_E) \to (Z', \mathfrak{A}_{E'}) \tag{3.3.6}$$

of simple graded manifolds seen as local-ringed spaces. This is the pair $(\zeta, \zeta_* \circ \Phi^*)$ of the morphism ζ of the body manifolds and the composition of the pull-back

$$\mathfrak{A}_{E'} \ni f \mapsto \Phi^* f \in \mathfrak{A}_E$$

of graded functions and the direct image ζ_* of the sheaf $\mathfrak{A}_E$ onto Z'. Relative to local bases (z^A, c^a) and (z'^A, c'^a) for $(Z, \mathfrak{A}_E)$ and $(Z', \mathfrak{A}_{E'})$ respectively, the morphism (3.3.6) reads

$$S\Phi(z) = \zeta(z), \qquad S\Phi(c'^a) = \Phi^a_b(z)c^b.$$

Accordingly, the pull-back onto Z of graded exterior forms on Z' is defined.

Given a graded manifold $(Z, \mathfrak{A})$, by the sheaf $\mathfrak{d}\mathfrak{A}$ of graded derivations of $\mathfrak{A}$ is meant a subsheaf of endomorphisms of the structure sheaf $\mathfrak{A}$ such that any section u of $\mathfrak{d}\mathfrak{A}$ over an open subset $U \subset Z$ is a *graded derivation* of the graded algebra $\mathfrak{A}(U)$. Conversely, one can show that, given open sets $U' \subset U$, there is a surjection of the derivation modules

$$\mathfrak{d}(\mathfrak{A}(U)) \to \mathfrak{d}(\mathfrak{A}(U'))$$

[5]. It follows that any graded derivation of the local graded algebra $\mathfrak{A}(U)$ is also a local section over U of the sheaf $\mathfrak{d}\mathfrak{A}$. Sections of $\mathfrak{d}\mathfrak{A}$ are called *graded vector fields* on the graded manifold $(Z, \mathfrak{A})$. They make up the graded derivation module $\mathfrak{d}\mathfrak{A}(Z)$ of the graded commutative $\mathbb{R}$-ring $\mathfrak{A}(Z)$. This module is a Lie superalgebra with respect to the superbracket (3.2.4).

In comparison with general theory of graded manifolds, an essential simplification is that graded vector fields on a simple graded manifold $(Z, \mathfrak{A}_E)$ can be seen as sections of a vector bundle as follows.

Due to the vertical splitting

$$VE \cong E \times E,$$

the vertical tangent bundle VE of $E \to Z$ can be provided with the fibre bases $\{\partial/\partial c^a\}$, which are the duals of the bases $\{c^a\}$. These are the fibre bases for

$$\mathrm{pr}_2 VE \cong E.$$

Then graded vector fields on a trivialization chart $(U; z^A, y^a)$ of E read

$$u = u^A \partial_A + u^a \frac{\partial}{\partial c^a}, \tag{3.3.7}$$

where u^λ, u^a are local graded functions on U. In particular,

$$\frac{\partial}{\partial c^a} \circ \frac{\partial}{\partial c^b} = -\frac{\partial}{\partial c^b} \circ \frac{\partial}{\partial c^a}, \qquad \partial_A \circ \frac{\partial}{\partial c^a} = \frac{\partial}{\partial c^a} \circ \partial_A.$$

The derivations (3.3.7) act on graded functions $f \in \mathfrak{A}_E(U)$ (3.3.3) by the rule

$$u(f_{a\ldots b}c^a \cdots c^b) = u^A \partial_A(f_{a\ldots b})c^a \cdots c^b + u^k f_{a\ldots b} \frac{\partial}{\partial c^k} \rfloor (c^a \cdots c^b). \tag{3.3.8}$$

This rule implies the corresponding coordinate transformation law

$$u'^A = u^A, \qquad u'^a = \rho^a_j u^j + u^A \partial_A(\rho^a_j) c^j$$

of graded vector fields. It follows that graded vector fields (3.3.7) can be represented by sections of the vector bundle $\mathcal{V}_E \to Z$ which is locally isomorphic to the vector bundle

$$\mathcal{V}_E|_U \approx \wedge E^* \underset{Z}{\otimes} (E \underset{Z}{\oplus} TZ)|_U, \tag{3.3.9}$$

and is characterized by an atlas of bundle coordinates

$$(z^A, z^A_{a_1\ldots a_k}, v^i_{b_1\ldots b_k}), \qquad k = 0, \ldots, m,$$

possessing the transition functions

$$z'^A_{i_1\ldots i_k} = \rho^{-1}{}^{a_1}_{i_1} \cdots \rho^{-1}{}^{a_k}_{i_k} z^A_{a_1\ldots a_k},$$
$$v'^i_{j_1\ldots j_k} = \rho^{-1}{}^{b_1}_{j_1} \cdots \rho^{-1}{}^{b_k}_{j_k} \left[\rho^i_j v^j_{b_1\ldots b_k} + \frac{k!}{(k-1)!} z^A_{b_1\ldots b_{k-1}} \partial_A \rho^i_{b_k} \right],$$

which fulfil the cocycle condition. Thus, the derivation module $\mathfrak{d}\mathfrak{A}_E(Z)$ is isomorphic to the structure module $\mathcal{V}_E(Z)$ of global sections of the vector bundle $\mathcal{V}_E \to Z$.

There is the exact sequence

$$0 \to \wedge E^* \underset{Z}{\otimes} E \to \mathcal{V}_E \to \wedge E^* \underset{Z}{\otimes} TZ \to 0 \tag{3.3.10}$$

of vector bundles over Z. Its splitting

$$\widetilde{\gamma} : \dot{z}^A \partial_A \mapsto \dot{z}^A (\partial_A + \widetilde{\gamma}^a_A \frac{\partial}{\partial c^a}) \tag{3.3.11}$$

transforms every vector field τ on Z into the graded vector field

$$\tau = \tau^A \partial_A \mapsto \nabla_\tau = \tau^A (\partial_A + \widetilde{\gamma}^a_A \frac{\partial}{\partial c^a}), \tag{3.3.12}$$

which is a graded derivation of the graded commutative $\mathbb{R}$-ring $\mathfrak{A}_E(Z)$ satisfying the Leibniz rule

$$\nabla_\tau (sf) = (\tau \rfloor ds) f + s \nabla_\tau(f), \quad f \in \mathfrak{A}_E(Z), \quad s \in C^\infty(Z).$$

It follows that the splitting (3.3.11) of the exact sequence (3.3.10) yields a connection on the graded commutative $C^\infty(Z)$-ring $\mathfrak{A}_E(Z)$ in accordance with Definition 3.2.2. It is called a *graded connection* on the simple graded manifold $(Z, \mathfrak{A}_E)$. In particular, this connection provides the corresponding horizontal splitting

$$u = u^A \partial_A + u^a \frac{\partial}{\partial c^a} = u^A (\partial_A + \widetilde{\gamma}^a_A \frac{\partial}{\partial c^a}) + (u^a - u^A \widetilde{\gamma}^a_A) \frac{\partial}{\partial c^a}$$

of graded vector fields. A graded connection (3.3.11) always exists [41].

Remark 3.3.3: By virtue of the isomorphism (3.3.2), any connection $\widetilde{\gamma}$ on a graded manifold $(Z,\mathfrak{A})$, restricted to a splitting domain U, takes the form (3.3.11). Given two splitting domains U and U' of $(Z,\mathfrak{A})$ with the transition functions (3.3.5), the connection components $\widetilde{\gamma}^a_A$ obey the transformation law

$$\widetilde{\gamma}'^a_A = \widetilde{\gamma}^b_A \frac{\partial}{\partial c^b}\rho^a + \partial_A\rho^a. \tag{3.3.13}$$

If U and U' are the trivialization charts of the same vector bundle E in Theorem 3.3.1 together with the transition functions (3.3.4), the transformation law (3.3.13) takes the form

$$\widetilde{\gamma}'^a_A = \rho^a_b(z)\widetilde{\gamma}^b_A + \partial_A\rho^a_b(z)c^b. \tag{3.3.14}$$

$\diamondsuit$

Remark 3.3.4: It should be emphasized that the above notion of a graded connection is a connection on the graded commutative ring $\mathfrak{A}_E(Z)$ seen as a $C^\infty(Z)$-module. It differs from that of a connection on a graded fibre bundle $(Z,\mathfrak{A})\to(X,\mathcal{B})$ in [1]. The latter is a connection on a graded $\mathcal{B}(X)$-module represented by a section of the jet graded bundle $J^1(Z/X)\to(Z,\mathfrak{A})$ of sections of the graded fibre bundle $(Z,\mathfrak{A})\to(X,\mathcal{B})$ [74]. $\diamondsuit$

Example 3.3.5: Every linear connection

$$\gamma = dz^A\otimes(\partial_A + \gamma_A{}^a{}_b y^b\partial_a)$$

on the vector bundle $E\to Z$ yields the graded connection

$$\gamma_S = dz^A\otimes(\partial_A + \gamma_A{}^a{}_b c^b\frac{\partial}{\partial c^a}) \tag{3.3.15}$$

on the simple graded manifold $(Z,\mathfrak{A}_E)$. In view of Remark 3.3.3, γ_S is also a graded connection on the graded manifold

$$(Z,\mathfrak{A})\cong(Z,\mathfrak{A}_E),$$

but its linear form (3.3.15) is not maintained under the transformation law (3.3.13). $\diamondsuit$

The *curvature* of the graded connection ∇_τ (3.3.12) is defined by the expression (1.6.16):

$$\begin{aligned}
&R(\tau,\tau') = [\nabla_\tau,\nabla_{\tau'}] - \nabla_{[\tau,\tau']},\\
&R(\tau,\tau') = \tau^A\tau'^B R^a_{AB}\frac{\partial}{\partial c^a} : \mathfrak{A}_E(Z)\to\mathfrak{A}_E(Z),\\
&R^a_{AB} = \partial_A\widetilde{\gamma}^a_B - \partial_B\widetilde{\gamma}^a_A + \widetilde{\gamma}^k_A\frac{\partial}{\partial c^k}\widetilde{\gamma}^a_B - \widetilde{\gamma}^k_B\frac{\partial}{\partial c^k}\widetilde{\gamma}^a_A.
\end{aligned} \tag{3.3.16}$$

It can also be written in the form

$$R : \mathfrak{A}_E(Z) \to \mathcal{O}^2(Z) \otimes \mathfrak{A}_E(Z),$$
$$R = \frac{1}{2} R^a_{AB} dz^A \wedge dz^B \otimes \frac{\partial}{\partial c^a}. \qquad (3.3.17)$$

Let now $\mathcal{V}^*_E \to Z$ be a vector bundle which is the pointwise $\wedge E^*$-dual of the vector bundle $\mathcal{V}_E \to Z$. It is locally isomorphic to the vector bundle

$$\mathcal{V}^*_E|_U \approx \wedge E^* \underset{Z}{\otimes} (E^* \underset{Z}{\oplus} T^*Z)|_U. \qquad (3.3.18)$$

With respect to the dual bases $\{dz^A\}$ for T^*Z and $\{dc^b\}$ for

$$\mathrm{pr}_2 V^*E \cong E^*,$$

sections of the vector bundle $\mathcal{V}^*_E$ take the coordinate form

$$\phi = \phi_A dz^A + \phi_a dc^a,$$

together with transition functions

$$\phi'_a = \rho^{-1b}{}_a \phi_b, \qquad \phi'_A = \phi_A + \rho^{-1b}{}_a \partial_A(\rho^a_j)\phi_b c^j.$$

They are regarded as *graded exterior one-forms* on the graded manifold $(Z, \mathfrak{A}_E)$, and make up the $\mathfrak{A}_E(Z)$-dual

$$\mathcal{C}^1_E = \mathfrak{d}\mathfrak{A}_E(Z)^*$$

of the derivation module

$$\mathfrak{d}\mathfrak{A}_E(Z) = \mathcal{V}_E(Z).$$

Conversely,

$$\mathfrak{d}\mathfrak{A}_E(Z) = (\mathcal{C}^1_E)^*.$$

The duality morphism is given by the graded interior product

$$u\rfloor\phi = u^A\phi_A + (-1)^{[\phi_a]} u^a \phi_a. \qquad (3.3.19)$$

In particular, the dual of the exact sequence (3.3.10) is the exact sequence

$$0 \to \wedge E^* \underset{Z}{\otimes} T^*Z \to \mathcal{V}^*_E \to \wedge E^* \underset{Z}{\otimes} E^* \to 0. \qquad (3.3.20)$$

Any graded connection $\widetilde{\gamma}$ (3.3.11) yields the splitting of the exact sequence (3.3.20), and determines the corresponding decomposition of graded one-forms

$$\phi = \phi_A dz^A + \phi_a dc^a = (\phi_A + \phi_a \widetilde{\gamma}^a_A) dz^A + \phi_a (dc^a - \widetilde{\gamma}^a_A dz^A).$$

Higher degree graded exterior forms are defined as sections of the exterior bundle $\overset{k}{\underset{Z}{\wedge}}\mathcal{V}_E^*$. They make up a bigraded algebra $\mathcal{C}_E^*$ which is isomorphic to the bigraded exterior algebra of the graded module $\mathcal{C}_E^1$ over $\mathcal{C}_E^0 = \mathfrak{A}(Z)$. This algebra is locally generated by graded forms dz^A, dc^i such that

$$dz^A \wedge dc^i = -dc^i \wedge dz^A, \qquad dc^i \wedge dc^j = dc^j \wedge dc^i. \tag{3.3.21}$$

The *graded exterior differential* d of graded functions is introduced by the condition $u\rfloor df = u(f)$ for an arbitrary graded vector field u, and is extended uniquely to graded exterior forms by the rule (3.2.17). It is given by the coordinate expression

$$d\phi = dz^A \wedge \partial_A \phi + dc^a \wedge \frac{\partial}{\partial c^a}\phi,$$

where the derivatives ∂_λ, $\partial/\partial c^a$ act on coefficients of graded exterior forms by the formula (3.3.8), and they are graded commutative with the graded forms dz^A and dc^a. The formulae (3.2.16) – (3.2.19) hold.

The graded exterior differential d makes $\mathcal{C}_E^*$ into a bigraded differential algebra whose de Rham complex reads

$$0 \to \mathbb{R} \to \mathfrak{A}_E(Z) \xrightarrow{d} \mathcal{C}_E^1 \xrightarrow{d} \cdots \mathcal{C}_E^k \xrightarrow{d} \cdots. \tag{3.3.22}$$

Its cohomology $H_{GR}^*(Z)$ is called the *graded de Rham cohomology* of the graded manifold $(Z, \mathfrak{A}_E)$. One can compute this cohomology with the aid of the abstract de Rham theorem. Let $\mathfrak{O}^k\mathfrak{A}_E$ denote the sheaf of germs of graded k-forms on $(Z, \mathfrak{A}_E)$. Its structure module is $\mathcal{C}_E^k$. These sheaves make up the complex

$$0 \to \mathbb{R} \longrightarrow \mathfrak{A}_E \xrightarrow{d} \mathfrak{O}^1\mathfrak{A}_E \xrightarrow{d} \cdots \mathfrak{O}^k\mathfrak{A}_E \xrightarrow{d} \cdots. \tag{3.3.23}$$

Its members $\mathfrak{O}^k\mathfrak{A}_E$ are sheaves of C_Z^∞-modules on Z and, consequently, are fine and acyclic. Furthermore, the Poincaré lemma for graded exterior forms holds [5]. It follows that the complex (3.3.23) is a fine resolution of the constant sheaf $\mathbb{R}$ on the manifold Z. Then, by virtue of Theorem 5.3.6, there is an isomorphism

$$H_{GR}^*(Z) = H^*(Z; \mathbb{R}) = H^*(Z) \tag{3.3.24}$$

of the graded de Rham cohomology $H_{GR}^*(Z)$ to the de Rham cohomology $H^*(Z)$ of the smooth manifold Z. Moreover, the cohomology isomorphism (3.3.24) accompanies the cochain monomorphism $\mathcal{O}^*(Z) \to \mathcal{C}_E^*$ of the de Rham complex $\mathcal{O}^*(Z)$ (1.6.4) of smooth exterior forms on Z to the graded de Rham complex (3.3.22). Hence, any closed graded exterior form is split into a sum $\phi = d\sigma + \varphi$ of an exact graded exterior form $d\sigma \in \mathcal{O}^*\mathfrak{A}_E$ and a closed exterior form $\varphi \in \mathcal{O}^*(Z)$ on Z.

3.4 Supermanifolds

There are different types of supermanifolds. These are H^∞-, G^∞-, GH^∞-, G-, and DeWitt supermanifolds [5, 41, 80]. By analogy with smooth manifolds, supermanifolds are constructed by gluing together of open subsets of supervector spaces $B^{n,m}$ with the aid of transition superfunctions. Therefore, let us start with the notion of a superfunction.

Though there are different classes of superfunctions, they can be introduced in the same manner as follows.

Let

$$B^{n,m} = \Lambda_0^n \oplus \Lambda_1^m, \qquad n,m \geq 0,$$

be a supervector space, where Λ is a Grassmann algebra of rank $0 < N \geq m$. Let

$$\sigma^{n,m} : B^{n,m} \to \mathbb{R}^n, \qquad s : B^{n,m} \to R^{n,m} = R_0^n \oplus R_1^m$$

be the corresponding body and soul maps (see the decomposition (3.1.2)). Then any element $q \in B^{n,m}$ is uniquely split as

$$q = (x,y) = (\sigma(x^i) + s(x^i))e_i^0 + y^j e_j^1, \tag{3.4.1}$$

where $\{e_i^0, e_j^1\}$ is a basis for $B^{n,m}$ and $\sigma(x^i) \in \mathbb{R}$, $s(x^i) \in R_0$, $y^j \in R_1$.

Let Λ' be another Grassmann algebra of rank $0 \leq N' \leq N$ which is treated as a subalgebra of Λ, i.e., the basis $\{c^a\}$, $a = 1,\dots,N'$, for Λ' is a subset of the basis $\{c^i\}$, $i = 1,\dots,N$, for Λ. Given an open subset $U \subset \mathbb{R}^n$, let us consider a Λ'-valued function

$$f(z) = \sum_{k=0}^{N'} \frac{1}{k!} f_{a_1\dots a_k}(z) c^{a_1} \cdots c^{a_k} \tag{3.4.2}$$

on U with smooth coefficients $f_{a_1\cdots a_k}(z)$, $z \in U$. It is a graded function on U. Its prolongation to $(\sigma^{n,0})^{-1}(U) \subset B^{n,0}$ is defined as the formal Taylor series

$$f(x) = \sum_{k=0}^{N'} \frac{1}{k!} \left[\sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^p f_{a_1\dots a_k}}{\partial z^{i_1} \cdots \partial z^{i_p}} (\sigma(x)) s(x^{i_1}) \cdots s(x^{i_p}) \right] c^{a_1} \cdots c^{a_k}. \tag{3.4.3}$$

Then a *superfunction* $F(q)$ on

$$(\sigma^{n,m})^{-1}(U) \subset B^{n,m}$$

is given by a sum

$$F(q) = F(x,y) = \sum_{r=0}^{m} \frac{1}{r!} f_{j_1\dots j_r}(x) y^{j_1} \cdots y^{j_r}, \tag{3.4.4}$$

where $f_{j_1\dots j_r}(x)$ are functions (3.4.3). However, the representation of a superfunction $F(x,y)$ by the sum (3.4.4) need not be unique.

The germs of superfunctions (3.4.4) constitute the sheaf $\mathfrak{S}_{N'}$ of graded commutative Λ'-algebras on $B^{n,m}$, but it is not a sheaf of $C^\infty_{B^{n,m}}$-modules since superfunctions are expressed in Taylor series.

Using the representation (3.4.4), one can define derivatives of superfunctions as follows. Let $f(x)$ be a superfunction on $B^{n,0}$. Since f, by definition, is the Taylor series (3.4.3), its partial derivative along an even coordinate x^i is defined in a natural way as

$$\partial_i f(x) = (\partial_i f)(\sigma(x), s(x)) = \tag{3.4.5}$$
$$\sum_{k=0}^{N'} \frac{1}{k!}\left[\sum_{p=0}^{N} \frac{1}{p!}\frac{\partial^{p+1} f_{a_1\ldots a_k}}{\partial z^i \partial z^{i_1}\cdots\partial z^{i_p}}(\sigma(x))s(x^{i_1})\cdots s(x^{i_p})\right] c^{a_1}\cdots c^{a_k}.$$

This even derivative is extended to superfunctions F on $B^{n,m}$ in spite of the fact that the representation (3.4.4) is not necessarily unique. However, the definition of odd derivatives of superfunctions is more intricate.

Let $\mathfrak{S}^0_{N'} \subset \mathfrak{S}_{N'}$ be the subsheaf of superfunctions $F(x,y) = f(x)$ (3.4.3) independent of the odd arguments y^j. Let $\wedge\mathbb{R}^m$ be a Grassmann algebra generated by $(a^1,\ldots,a^m)$. The expression (3.4.4) implies that, for any open subset $U \subset B^{n,m}$, there exists the sheaf morphism

$$\lambda : \mathfrak{S}^0_{N'} \otimes \wedge\mathbb{R}^m \to \mathfrak{S}_{N'}, \tag{3.4.6}$$

$$\lambda(x,y) : \sum_{r=0}^{m} \frac{1}{r!} f_{j_1\ldots j_r}(x) \otimes (a^{j_1}\cdots a^{j_r}) \to \tag{3.4.7}$$
$$\sum_{r=0}^{m} \frac{1}{r!} f_{j_1\ldots j_r}(x) y^{j_1}\cdots y^{j_r},$$

over $B^{n,m}$. Clearly, the morphism λ (3.4.6) is an epimorphism. One can show that this epimorphism is injective and, consequently, is an isomorphism iff

$$N - N' \geq m \tag{3.4.8}$$

[5]. Roughly speaking, in this case, there exists a tuple of elements $y^{j_1},\ldots,y^{j_r} \in \Lambda$ for each superfunction f such that

$$\lambda(f \otimes (a^{j_1}\cdots a^{j_r})) \neq 0$$

at the point $(x, y^{j_1},\ldots,y^{j_m})$ of $B^{n,m}$.

If the condition (3.4.8) holds, the representation of each superfunction $F(x,y)$ by the sum (3.4.4) is unique, and it is an image of some section $f \otimes a$ of the sheaf $\mathfrak{S}^0_{N'} \otimes \wedge\mathbb{R}^m$ with respect to the morphism λ (3.4.7). Then an odd derivative of F is defined as

$$\frac{\partial}{\partial y^j}(\lambda(f \otimes y)) = \lambda(f \otimes \frac{\partial}{\partial a^j}(a)).$$

This definition is consistent only if λ is an isomorphism, i.e., the relation (3.4.8) holds. If otherwise, there exists a non-vanishing element $f \otimes a$ such that

$$\lambda(f \otimes a) = 0,$$

whereas

$$\lambda(f \otimes \partial_j(a)) \neq 0.$$

For instance, if

$$N - N' = m - 1,$$

such an element is

$$f \otimes a = c^1 \cdots c^{N'} \otimes (a^1 \cdots a^m).$$

In order to classify superfunctions, we follow the terminology of [5, 72, 73].

• If $N' = N$, one deals with G^∞-*superfunctions*, introduced in [71]. In this case, the inequality (3.4.8) is not satisfied, unless $m = 0$.

• If the condition (3.4.8) holds, $\mathfrak{S}_{N'} = \mathcal{GH}_{N'}$ is the sheaf of GH^∞-*superfunctions*.

• In particular, if $N' = 0$, the condition (3.4.8) is satisfied, and $\mathfrak{S}_{N'} = \mathcal{H}^\infty$ is the sheaf of H^∞-*superfunctions*

$$F(x,y) = \sum_{r=0}^{m} \frac{1}{r!}\left[\sum_{p=0}^{N} \frac{1}{p!}\frac{\partial^p f_{j_1\ldots j_r}}{\partial z^{i_1}\cdots\partial z^{i_p}}(\sigma(x))s(x^{i_1})\cdots s(x^{i_p})\right] y^{j_1}\cdots y^{j_r}, \tag{3.4.9}$$

where $f_{j_1\ldots j_r}$ are real functions [7, 30].

Superfunctions of the above three types are called *smooth superfunctions*. The fourth type of superfunctions is the following.

Given the sheaf $\mathcal{GH}_{N'}$ of GH^∞-superfunctions on a supervector space $B^{n,m}$, let us define the sheaf of graded commutative Λ-algebras

$$\mathcal{G}_{N'} = \mathcal{GH}_{N'} \underset{\Lambda'}{\otimes} \Lambda, \tag{3.4.10}$$

where Λ is regarded as a graded commutative Λ'-algebra. The sheaf $\mathcal{G}_{N'}$ (3.4.10) possesses the following important properties [5].

• There is the *evaluation morphism*

$$\delta : \mathcal{G}_{N'} \ni F \otimes a \mapsto Fa \in C^\Lambda_{B^{n,m}}, \tag{3.4.11}$$

where

$$C^\Lambda_{B^{n,m}} \cong C^0_{B^{n,m}} \otimes \Lambda$$

is the sheaf of continuous Λ-valued functions on $B^{n,m}$. Its image is isomorphic to the sheaf $\mathcal{G}^\infty$ of G^∞-superfunctions on $B^{n,m}$.

• For any two integers N' and N'' satisfying the condition (3.4.8), there exists the canonical isomorphism between the sheaves $\mathcal{G}_{N'}$ and $\mathcal{G}_{N''}$. Therefore, one can define the canonical sheaf $\mathcal{G}_{n,m}$ of graded commutative Λ-algebras on the supervector space $B^{n,m}$ whose sections can be seen as tensor products $F\otimes a$ of H^∞-superfunctions F (3.4.9) and elements $a\in\Lambda$. They are called *G-superfunctions*.

• The sheaf $\mathfrak{d}\mathcal{G}_{n,m}$ of graded derivations of the sheaf $\mathcal{G}_{n,m}$ is a locally free sheaf of $\mathcal{G}_{n,m}$-modules of rank (n,m). On any open set $U\subset B^{n,m}$, the $\mathcal{G}_{n,m}(U)$-module $\mathfrak{d}\mathcal{G}_{n,m}(U)$ is generated by the derivations $\partial/\partial x^i$, $\partial/\partial y^j$ which act on $\mathcal{G}_{n,m}(U)$ by the rule

$$\frac{\partial}{\partial x^i}(F\otimes a)=\frac{\partial F}{\partial x^i}\otimes a,\qquad \frac{\partial}{\partial y^j}(F\otimes a)=\frac{\partial F}{\partial y^j}\otimes a. \tag{3.4.12}$$

These properties of G-superfunctions make G-supermanifolds most suitable for differential geometric constructions.

A paracompact topological space M is said to be an (n,m)-dimensional *smooth supermanifold* if it admits an atlas

$$\Psi=\{U_\zeta,\phi_\zeta\},\qquad \phi_\zeta:U_\zeta\to B^{n,m},$$

such that the transition functions $\phi_\zeta\circ\phi_\xi^{-1}$ are supersmooth. Obviously, a smooth supermanifold of dimension (n,m) is also a real smooth manifold of dimension $2^{N-1}(n+m)$. If transition superfunctions are H^∞-, G^∞- or GH^∞-superfunctions, one deals with H^∞-, G^∞- or GH^∞-supermanifolds, respectively.

By virtue of Theorem 1.6.1 extended to graded local-ringed spaces, this definition is equivalent to the following one.

DEFINITION 3.4.1: A smooth supermanifold is a graded local-ringed space $(M,\mathfrak{S})$ which is locally isomorphic to $(B^{n,m},\mathcal{S})$, where $\mathcal{S}$ is one of the sheaves of smooth superfunctions on $B^{n,m}$. The sheaf $\mathcal{S}$ is called the *structure sheaf* of a smooth supermanifold. □

In accordance with Definition 3.4.1, by a morphism of smooth supermanifolds is meant their morphism (φ,Φ) as graded local-ringed spaces, where Φ is an even graded morphism. In particular, every morphism $\varphi:M\to M'$ yields the smooth supermanifold morphism $(\varphi,\Phi=\varphi^*)$.

Smooth supermanifolds however are effected by serious inconsistencies as follows. Since odd derivatives of G^∞-superfunctions are ill defined, the sheaf of derivations of the sheaf of G^∞-superfunctions is not locally free. Nevertheless, any G-supermanifold has an underlying G^∞-supermanifold.

In the case of GH^∞-supermanifolds (including H^∞-ones), spaces of values of GH^∞-superfunctions at different points are not mutually isomorphic because the Grassmann algebra Λ is not a free module with respect to its subalgebra Λ'. By these reasons, we turn to G-supermanifolds. Their definition repeats Definition 3.4.1.

DEFINITION 3.4.2: An (n,m)-dimensional *G-supermanifold* is a graded local-ringed space (M, G_M), satisfying the following conditions:

• M is a paracompact topological space;

• (M, G_M) is locally isomorphic to $(B^{n,m}, \mathcal{G}_{n,m})$;

• there exists a morphism of sheaves of graded commutative Λ-algebras $\delta : G_M \to C^\Lambda_M$, where

$$C^\Lambda_M \cong C^0_M \otimes \Lambda$$

is sheaf of continuous Λ-valued functions on M, and δ is locally isomorphic to the evaluation morphism (3.4.11). □

Example 3.4.1: The triple $(B^{n,m}, \mathcal{G}_{n,m}, \delta)$, where δ is the evaluation morphism (3.4.11), is called the *standard G-supermanifold*. For any open subset $U \subset B^{n,m}$, the space $\mathcal{G}_{n,m}(U)$ can be provided with the topology which makes it into a graded Fréchet algebra. Then there are isometrical isomorphisms

$$\mathcal{G}_{n,m}(U) \cong \mathcal{H}^\infty(U) \otimes \Lambda \cong C^\infty(\sigma^{n,m}(U)) \otimes \Lambda \otimes \wedge\mathbb{R}^m \cong \\ C^\infty(\sigma^{n,m}(U)) \otimes \wedge\mathbb{R}^{N+m}. \tag{3.4.13}$$

♢

Remark 3.4.2: Any GH^∞-supermanifold (M, GH^∞_M) with the structure sheaf GH^∞_M is naturally extended to the G-supermanifold $(M, GH^\infty_M \otimes \Lambda)$. Every G-supermanifold defines an *underlying G^∞-supermanifold* $(M, \delta(G_M))$, where $\delta(G_M) = G^\infty_M$ is the sheaf of G^∞-superfunctions on M. ♢

As in the case of smooth supermanifolds, the underlying space M of a G-supermanifold (M, G_M) is provided with the structure of a real smooth manifold of dimension $2^{N-1}(n+m)$, and morphisms of G-supermanifolds are smooth morphisms of the underlying smooth manifolds. However, it may happen that non-isomorphic G-supermanifold have isomorphic underlying smooth manifolds.

Let (M, G_M) be a G-supermanifold. Sections u of the sheaf $\mathfrak{d}G_M$ of graded derivations are called *supervector fields* on the G-supermanifold (M, G_M), while sections ϕ of the dual sheaf $\mathfrak{d}G^*_M$ are *one-superforms* on (M, G_M). Given a coordinate chart $(q^i) = (x^i, y^j)$ on $U \subset M$, supervector fields and one-superforms read

$$u = u^i \partial_i, \qquad \phi = \phi_i dq^i,$$

where coefficients u^i and ϕ_i are G-superfunctions on U. The graded differential calculus in supervector fields and superforms obeys the standard formulae (3.2.4), (3.2.16), (3.2.17) and (3.2.18).

Let us consider cohomology of G-supermanifolds. Given a G-supermanifold (M, G_M), let

$$\mathfrak{O}^k_{\Lambda M} = \mathfrak{O}^k_M \otimes \Lambda$$

be the sheaves of smooth Λ-valued exterior forms on M. These sheaves are fine, and they constitute the fine resolution

$$0 \to \Lambda \to C^\infty_M \otimes \Lambda \to \mathfrak{O}^1_M \otimes \Lambda \to \cdots$$

of the constant sheaf Λ on M. We have the corresponding de Rham complex

$$0 \to \Lambda \to C^\infty_\Lambda(M) \to \mathcal{O}^1_\Lambda(M) \to \cdots$$

of Λ-valued exterior forms on M. By virtue of Theorem 5.3.6, the cohomology $H^*_\Lambda(M)$ of this complex is isomorphic to the sheaf cohomology $H^*(M;\Lambda)$ of M with coefficients in the constant sheaf Λ and, consequently, is related to the de Rham cohomology as follows:

$$H^*_\Lambda(M) = H^*(M;\Lambda) = H^*(M) \otimes \Lambda. \tag{3.4.14}$$

Thus, the cohomology groups of Λ-valued exterior forms do not provide us with information on the G-supermanifold structure of M.

Let us turn to cohomology of superforms on a G-supermanifold (M, G_M). The sheaves $\overset{k}{\wedge}\mathfrak{d}G^*_M$ of superforms constitute the complex

$$0 \to \Lambda \to G_M \to \mathfrak{d}^*G_M \to \cdots. \tag{3.4.15}$$

The Poincaré lemma for superforms is proved to hold [5, 12], and this complex is exact. However, the structure sheaf G_M need not be acyclic, and the exact sequence (3.4.15) fails to be a resolution of the constant sheaf Λ on M in general. Therefore, the cohomology $H^*_S(M)$ of the de Rham complex of superforms are not equal to cohomology $H^*(M;\Lambda)$ of M with coefficients in the constant sheaf Λ, and need not be related to the de Rham cohomology $H^*(M)$ of the smooth manifold M. In particular, cohomology $H^*_S(M)$ is not a topological invariant, but it is invariant under G-isomorphisms of G-supermanifolds.

PROPOSITION 3.4.3: The structure sheaf $\mathcal{G}_{n,m}$ of the standard G-supermanifold $(B^{n,m}, \mathcal{G}_{n,m})$ is acyclic, i.e.,

$$H^{k>0}(B^{n,m};\mathcal{G}_{n,m}) = 0.$$

□

The proof is based on the isomorphism (3.4.13) and some cohomological constructions [5, 13].

3.5 Supervector bundles

As was mentioned above, supervector bundles are considered in the category of G-supermanifolds. We start with the definition of the product of two G-supermanifolds seen as a trivial supervector bundle.

Let $(B^{n,m}, \mathcal{G}_{n,m})$ and $(B^{r,s}, \mathcal{G}_{r,s})$ be two standard G-supermanifolds in Example 3.4.2. Given open sets $U \subset B^{n,m}$ and $V \subset B^{r,s}$, we consider the presheaf

$$U \times V \to \mathcal{G}_{n,m}(U)\widehat{\otimes}\mathcal{G}_{r,s}(V), \tag{3.5.1}$$

where $\widehat{\otimes}$ denotes the tensor product of modules completed in Grothendieck's topology (see Remark 1.6.2). Due to the isomorphism (3.4.13), it is readily observed that the structure sheaf $\mathcal{G}_{n+r,m+s}$ of the standard G-supermanifold on $B^{n+r,m+s}$ is isomorphic to that, defined by the presheaf (3.5.1). This construction is generalized to arbitrary G-supermanifolds as follows.

Let (M, G_M) and $(M', G_{M'})$ be two G-supermanifolds of dimensions (n, m) and (r, s), respectively. Their *product*

$$(M, G_M) \times (M', G_{M'})$$

is defined as the graded local-ringed space $(M \times M', G_M\widehat{\otimes}G_{M'})$, where $G_M\widehat{\otimes}G_{M'}$ is the sheaf determined by the presheaf

$$\begin{aligned}
&U \times U' \to G_M(U)\widehat{\otimes}G_{M'}(U'),\\
&\delta : G_M(U)\widehat{\otimes}G_{M'}(U') \to C^\infty_{\sigma(U)}\widehat{\otimes}C^\infty_{\sigma(U')} = C^\infty_{\sigma_M(U)\times\sigma_M(U')},
\end{aligned}$$

for any open subsets $U \subset M$ and $U' \subset M'$. This product is a G-supermanifold of dimension $(n + r, m + s)$ [5]. Furthermore, there is the epimorphism

$$\mathrm{pr}_1 : (M, G_M) \times (M', G_{M'}) \to (M, G_M).$$

One may define its section over an open subset $U \subset M$ as the G-supermanifold morphism

$$s_U : (U, G_M|_U) \to (M, G_M) \times (M', G_{M'})$$

such that $\mathrm{pr}_1 \circ s_U$ is the identity morphism of $(U, G_M|_U)$. Sections s_U over all open subsets $U \subset M$ determine a sheaf on M. This sheaf should be provided with a suitable graded commutative G_M-structure.

For this purpose, let us consider the product

$$(M, G_M) \times (B^{r|s}, \mathcal{G}_{r|s}), \tag{3.5.2}$$

where $B^{r|s}$ is the superspace (3.1.4). It is called a *product G-supermanifold*. Since the Λ_0-modules $B^{r|s}$ and $B^{r+s,r+s}$ are isomorphic, $B^{r|s}$ has a natural structure of an $(r+s, r+s)$-dimensional G-supermanifold. Because $B^{r|s}$ is a free graded Λ-module of the type (r, s), the sheaf $S^{r|s}_M$ of sections of the fibration

$$(M, G_M) \times (B^{r|s}, \mathcal{G}_{r|s}) \to (M, G_M) \tag{3.5.3}$$

has the structure of the sheaf of free graded G_M-modules of rank (r, s). Conversely, given a G-supermanifold (M, G_M) and a sheaf S of free graded G_M-modules of rank (r, s) on

M, there exists a product G-supermanifold (3.5.2) such that S is isomorphic to the sheaf of sections of the fibration (3.5.3).

Let us turn now to the notion of a supervector bundle over G-supermanifolds. Similarly to smooth vector bundles (see Theorem 1.6.3), one can require of the category of supervector bundles over G-supermanifolds to be equivalent to the category of locally free sheaves of graded modules on G-supermanifolds. Therefore, we can restrict ourselves to locally trivial supervector bundles with the standard fibre $B^{r|s}$.

DEFINITION 3.5.1: A *supervector bundle* over a G-supermanifold (M, G_M) with the standard fibre $(B^{r|s}, \mathcal{G}_{r|s})$ is defined as a pair $((Y, G_Y), \pi)$ of a G-supermanifold (Y, G_Y) and a G-epimorphism

$$\pi : (Y, G_Y) \to (M, G_M) \tag{3.5.4}$$

such that M admits an atlas $\{(U_\zeta, \psi_\zeta\}$ of local G-isomorphisms

$$\psi_\zeta : (\pi^{-1}(U_\zeta), G_Y|_{\pi^{-1}(U_\zeta)}) \to (U_\zeta, G_M|_{U_\zeta}) \times (B^{r|s}, \mathcal{G}_{r|s}).$$

□

It is clear that sections of the supervector bundle (3.5.4) constitute a locally free sheaf of graded G_M-modules. The converse of this fact is the following [5].

THEOREM 3.5.2: For any locally free sheaf S of graded G_M-modules of rank (r, s) on a G-supermanifold (M, G_M), there exists a supervector bundle over (M, G_M) such that S is isomorphic to the structure sheaf of its sections. □

The fibre Y_q, $q \in M$, of the supervector bundle in Theorem 3.5.2 is the quotient

$$S_q/\mathcal{M}_q \cong S^{r|s}_{Mq}/(\mathcal{M}_q \cdot S^{r|s}_{Mq}) \cong B^{r|s}$$

of the stalk S_q by the submodule $\mathcal{M}_q$ of the germs $s \in S_q$ whose evaluation $\delta(f)(q)$ vanishes. This fibre is a graded Λ-module isomorphic to $B^{r|s}$, and is provided with the structure of the standard G-supermanifold.

Remark 3.5.1: The proof of Theorem 3.5.2 is based on the fact that, given the transition functions $\rho_{\zeta\xi}$ of the sheaf S, their evaluations

$$g_{\zeta\xi} = \delta(\rho_{\zeta\xi}) \tag{3.5.5}$$

define the morphisms

$$U_\zeta \cap U_\xi \to GL(r|s; \Lambda),$$

and they are assembled into a cocycle of G^∞-morphisms from M to the general linear graded group $GL(r|s; \Lambda)$. Thus, we come to the notion of a G^∞*-vector bundle*. Its definition is a repetition of Definition 3.5.1 if one replaces G-supermanifolds and G-morphisms with the G^∞-ones. Moreover, the G^∞-supermanifold underlying a supervector bundle (see Remark 3.4.2) is

a G^∞-supervector bundle, whose transition functions $g_{\zeta\xi}$ are related to those of the supervector bundle by the evaluation morphisms (3.5.5), and are $GL(r|s;\Lambda)$-valued transition functions. $\diamondsuit$

Since the category of supervector bundles over a G-supermanifold (M, G_M) is equivalent to the category of locally free sheaves of graded G_M-modules, one can define the usual operations of direct sum, tensor product, etc. of supervector bundles.

Let us note that any supervector bundle admits the canonical global zero section. Any section of the supervector bundle π (3.5.4), restricted to its trivialization chart

$$(U, G_M\mid_U) \times (B^{r|s}, \mathcal{G}_{r|s}), \tag{3.5.6}$$

is represented by a sum $s = s^a(q)\epsilon_a$, where $\{\epsilon_a\}$ is the basis for the graded Λ-module $B^{r|s}$, while $s^a(q)$ are G-superfunctions on U. Given another trivialization chart U' of π, a transition function

$$s'^b(q)\epsilon'_b = s^a(q)h^b{}_a(q)\epsilon_b, \qquad q \in U \cap U', \tag{3.5.7}$$

is given by the $(r+s)\times(r+s)$ matrix h whose entries $h^b{}_a(q)$ are G-superfunctions on $U\cap U'$. One can think of this matrix as being a section of the supervector bundle over $U\cap U$ with the above mentioned group $GL(r|s;\Lambda)$ as a typical fibre.

Example 3.5.2: Given a G-supermanifold (M, G_M), let us consider the locally free sheaf $\mathfrak{d}G_M$ of graded derivations of G_M. In accordance with Theorem 3.5.2, there is a supervector bundle $T(M, G_M)$, called *supertangent bundle*, whose structure sheaf is isomorphic to $\mathfrak{d}G_M$. If $(q^1,\ldots,q^{m+n})$ and $(q'^1,\ldots,q'^{m+n})$ are two coordinate charts on M, the Jacobian matrix

$$h^i_j = \frac{\partial q'^i}{\partial q^j}, \qquad i,j = 1,\ldots,n+m,$$

(see the prescription (3.4.12)) provides the transition morphisms for $T(M, G_M)$.

It should be emphasized that the underlying G^∞-vector bundle of the supertangent bundle $T(M, G_M)$, called G^∞*-supertangent bundle*, has the transition functions $\delta(h^i_j)$ which cannot be written as the Jacobian matrices since the derivatives of G^∞-superfunctions with respect to odd arguments are ill defined and the sheaf $\mathfrak{d}G^\infty_M$ is not locally free. $\diamondsuit$

3.6 Superconnections

Given a supervector bundle π (3.5.4) with the structure sheaf S, one can follow suit of Definition 1.7.3 and introduce a connection on this supervector bundle as a splitting of the the exact sequence of sheaves

$$0 \to \mathfrak{d}G^*_M \otimes S \to (G_M \oplus \mathfrak{d}G^*_M)\otimes S \to S \to 0. \tag{3.6.1}$$

Its splitting is an even sheaf morphism

$$\nabla : S \to \mathfrak{d}^*G_M \otimes S \tag{3.6.2}$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s), \qquad f \in G_M(U), \qquad s \in S(U), \tag{3.6.3}$$

for any open subset $U \in M$. The sheaf morphism (3.6.2) is called a *superconnection* on the supervector bundle π (3.5.4). Its *curvature* is given by the expression

$$R = \nabla^2 : S \to \overset{2}{\wedge} \mathfrak{d} G_M^* \otimes S, \tag{3.6.4}$$

similar to the expression (1.7.4).

The exact sequence (3.6.1) need not be split. One can apply the criterion in Section 1.7 in order to study the existence of a superconnection on supervector bundles. Namely, the exact sequence (3.6.1) leads to the exact sequence of sheaves

$$0 \to \mathrm{Hom}\,(S, \mathfrak{d}G_M^* \otimes S) \to \mathrm{Hom}\,(S, (G_M \oplus \mathfrak{d}G_M^*) \otimes S) \to \mathrm{Hom}\,(S, S) \to 0$$

and to the corresponding exact sequence of the cohomology groups

$$\begin{aligned} 0 \to H^0(M; \mathrm{Hom}\,(S, \mathfrak{d}G_M^* \otimes S)) &\to H^0(M; \mathrm{Hom}\,(S, (G_M \oplus \mathfrak{d}G_M^*) \otimes S)) \\ &\to H^0(M; \mathrm{Hom}\,(S, S)) \to H^1(M; \mathrm{Hom}\,(S, \mathfrak{d}G_M^* \otimes S)) \to \cdots. \end{aligned}$$

The exact sequence (3.6.1) defines the Atiyah class

$$\mathrm{At}(\pi) \in H^1(M; \mathrm{Hom}\,(S, \mathfrak{d}G_M^* \otimes S))$$

of the supervector bundle π (3.5.4). If the Atiyah class vanishes, a superconnection on this supervector bundle exists. In particular, a superconnection exists if the cohomology set $H^1(M; \mathrm{Hom}\,(S, \mathfrak{d}G_M^* \otimes S))$ is trivial. In contrast with the sheaf of smooth functions, the structure sheaf G_M of a G-supermanifold is not acyclic in general, cohomology $H^*(M; \mathrm{Hom}\,(S, \mathfrak{d}G_M^* \otimes S))$ is not trivial, and a supervector bundle need not admit a superconnection.

Example 3.6.1: In accordance with Proposition 3.4.3, the structure sheaf of the standard G-supermanifold $(B^{n,m}, \mathcal{G}_{n,m})$ is acyclic, and the trivial supervector bundle

$$(B^{n,m}, \mathcal{G}_{n,m}) \times (B^{r|s}, \mathcal{G}_{r|s}) \to (B^{n,m}, \mathcal{G}_{n,m}) \tag{3.6.5}$$

has obviously a superconnection, e.g., the trivial superconnection. $\diamondsuit$

Example 3.6.1 enables one to obtain a local coordinate expression for a superconnection on a supervector bundle π (3.5.4), whose typical fibre is $B^{r|s}$ and whose base is a G-supermanifold locally isomorphic to the standard G-supermanifold $(B^{n,m}, \mathcal{G}_{n,m})$. Let $U \subset M$ (3.5.6) be a trivialization chart of this supervector bundle such that every section s of $\pi|_U$ is represented by a sum $s^a(q)\epsilon_a$, while the sheaf of one-superforms $\mathfrak{d}^* G_M|_U$ has a

local basis $\{dq^i\}$. Then a superconnection ∇ (3.6.2) restricted to this trivialization chart can be given by a collection of coefficients $\nabla_i{}^a{}_b$:

$$\nabla(\epsilon_a) = dq^i \otimes (\nabla_i{}^b{}_a \epsilon_b), \tag{3.6.6}$$

which are G-superfunctions on U. Bearing in mind the Leibniz rule (3.6.3), one can compute the coefficients of the curvature form (3.6.4) of the superconnection (3.6.6). We have

$$\begin{aligned} &R(\epsilon_a) = \frac{1}{2} dq^i \wedge dq^j \otimes R_{ij}{}^b{}_a \epsilon_b, \\ &R_{ij}{}^a{}_b = (-1)^{[i][j]} \partial_i \nabla_j{}^a{}_b - \partial_j \nabla_i{}^a{}_b + (-1)^{[i]([j]+[a]+[k])} \nabla_j{}^a{}_k \nabla_i{}^k{}_b - \\ &\qquad (-1)^{[j]([a]+[k])} \nabla_i{}^a{}_k \nabla_j{}^k{}_b. \end{aligned}$$

In a similar way, one can obtain the transformation law of the superconnection coefficients (3.6.6) under the transition morphisms (3.5.7). In particular, any trivial supervector bundle admits the trivial superconnection $\nabla_i{}^b{}_a = 0$.

Chapter 4

Non-commutative geometry

Non-commutative geometry is developed in main as a generalization of the calculus in commutative rings of smooth functions [23, 37, 41, 43, 53, 58]. Accordingly, a non-commutative generalization of differential geometry is phrased in terms of the differential calculus over a non-commutative ring which replaces the exterior algebra of differential forms. The Chevalley–Eilenberg differential calculus over a commutative ring is straightforwardly generalized to a non-commutative $\mathcal{K}$-ring $\mathcal{A}$. However, the extension of the notion of a differential operator to modules over a non-commutative ring meets difficulties [41, 77]. In a general setting, any non-commutative ring can be called into play, but one often follows the more deep analogy to the case of commutative smooth function rings. In Connes' commutative geometry, $\mathcal{A}$ is the algebra $\mathbb{C}^\infty(X)$ of smooth complex functions on a compact manifold X. It is a dense subalgebra of the C^*-algebra of continuous complex functions on X. Generalizing this case, Connes' non-commutative geometry [22, 23, 25] addresses the differential calculus over an involutive algebra $\mathcal{A}$ of bounded operators in a Hilbert space E and, furthermore, studies a representation of this differential calculus by operators in E.

4.1 Modules over C^*-algebras

Let us point out some features of modules over non-commutative algebras and, in particular, C^*-algebras.

Let $\mathcal{K}$ throughout be a commutative ring and $\mathcal{A}$ a $\mathcal{K}$-ring which need not be commutative. Let $\mathcal{Z}_\mathcal{A}$ denote its *center*. An $\mathcal{A}$-bimodule throughout is assumed to be a commutative (central) $\mathcal{Z}_\mathcal{A}$-bimodule. Sometimes, it is convenient to use the following compact abbreviation [35]. We say that right and left $\mathcal{A}$-modules, $\mathcal{A}$-bimodules and $\mathcal{Z}_\mathcal{A}$-bimodules are $\mathcal{A}_i$-modules of type $(1,0)$, $(0,1)$, $(1,1)$ and $(0,0)$, respectively (or $(\mathcal{A}_i - \mathcal{A}_j)$-modules where $\mathcal{A}_0 = \mathcal{Z}_\mathcal{A}$ and $\mathcal{A}_1 = \mathcal{A}$). Of course, $\mathcal{A}_i$-modules of type $(1,1)$ are also of type $(1,0)$ and $(0,1)$, while $\mathcal{A}_i$-modules of type $(1,0)$, $(0,1)$ $(1,1)$ are also of type $(0,0)$. With this abbreviation, the basic constructions of new modules from old ones are phrased as follows.

• If P and P' are $\mathcal{A}_i$-modules of the same type (i,j), so is its direct sum $P\oplus P'$.

• Let P and P' be $\mathcal{A}_i$-modules of type (i,k) and (k,j), respectively. Their tensor product $P\otimes P'$ is an $\mathcal{A}_i$-module of type (i,j).

• Given an $\mathcal{A}_i$-module P of type (i,j), its $\mathcal{A}$-dual $P^*=\mathrm{Hom}_{\mathcal{A}_i-\mathcal{A}_j}(P,\mathcal{A})$ is a module of type $(i+1,j+1)\mathrm{mod}\,2$.

Let A be a complex involutive algebra. Any module over A is also a complex vector space. An A_i-module of type $(1,1)$ is called an *involutive module* if it is equipped with an antilinear involution $p\mapsto p^*$ such that

$$(apb)^*=b^*p^*a^*, \qquad a,b\in A, \qquad p\in P.$$

Due to this relation, an involutive module is reconstructed by its right or left module structure. In particular, an involutive module is said to be a projective module of finite rank if, seen as a right (or left) module, it is a finite projective module.

Given a right module P over an involutive algebra A, a *Hermitian form* on P is defined as a sesquilinear A-valued form

$$\begin{aligned}&\langle .|.\rangle : P\times P\to A, \\ &\langle pa|p'a'\rangle = a^*\langle p|p'\rangle a', \qquad \langle p|p'\rangle=\langle p'|p\rangle^*, \qquad p,p'\in P, \quad a,a'\in A.\end{aligned} \tag{4.1.1}$$

A Hermitian form (4.1.1) on P yields an antilinear morphism h of P to its A-dual P^* given by the formula

$$(hp)(p')=\langle p|p'\rangle, \qquad p,p'\in P. \tag{4.1.2}$$

A Hermitian form (4.1.1) is called *invertible* if the morphism h is invertible.

Let A be a C^*-algebra. A Hermitian form (4.1.1) on a right A-module P is called *positive* if $\langle p|p\rangle$ for all $p\in P$ is a *positive element* of a C^*-algebra A, i.e.,

$$\langle p|p\rangle = aa^*, \qquad a\in A.$$

Let A be a unital C^*-algebra. Any projective A-module P of finite rank admits an invertible positive Hermitian form. Moreover, all these forms on P are isomorphic [63].

A positive Hermitian form on a right A-module P endows P with the semi-norm

$$||p|| = ||\langle p|p\rangle||^{1/2}, \qquad p\in P, \tag{4.1.3}$$

where $||\langle p|p\rangle||$ is the C^*-algebra norm of $\langle p|p\rangle\in A$. Equipped with this seminorm and the corresponding topology, P is called the (right) *pre-Hilbert module*. It is a *Hilbert module* (a *C^*-module* in the terminology of [23]) if the seminorm (4.1.3) is a complete norm.

Example 4.1.1: A C^*-algebra A is provided with the structure of a Hilbert A-module with respect to the action of A on itself by right multiplications and the positive Hermitian form

$$\langle a|a'\rangle = a^*a', \qquad a,a'\in A. \tag{4.1.4}$$

$\diamondsuit$

Example 4.1.2: Let $A = \mathbb{C}^0(X)$ be the C^*-algebra of continuous complex functions on a compact space X, and let $E \to X$ be a (topological) complex vector bundle endowed with a Hermitian fibre metric $\langle .|.\rangle_x$. Then the space $E(X)$ of continuous sections of $E \to X$ is a Hilbert $\mathbb{C}^0(X)$-module with respect to the $\mathbb{C}^0(X)$-valued Hermitian form

$$\langle s|s'\rangle(x) = \langle s(x)|s'(x)\rangle_x, \qquad s, s' \in E(X).$$

$\diamondsuit$

Given a Hilbert A-module P, by its *endomorphism* T is meant a continuous A-linear endomorphism of a right module P which admits the *adjoint endomorphism* T^*, which is uniquely given by the relation

$$\langle p|Tp'\rangle = \langle T^*p|p'\rangle, \qquad p, p' \in P.$$

The set $B_A(P)$ of A-linear endomorphisms of a Hilbert A-module P is a C^*-algebra with respect to the operator norm. *Compact endomorphisms* of P are defined as the closer of its endomorphisms of finite rank. Let us consider endomorphisms $T_{p,q} \in B_A(P)$ of P of the form

$$T_{p,q}p' = p\langle p'|q\rangle, \qquad p, p', q \in P. \tag{4.1.5}$$

They obey the relations

$$T^*_{p,q} = T_{q,p}, \qquad T_{p,q}T_{p',q'} = T_{p\langle q|p'\rangle,q'} = T_{p,T_{q,p'}q}.$$

The linear span of endomorphisms (4.1.5) is a two-sided ideal of $B_A(P)$ [68]. Its closure is the set $T_A(P)$ of compact A-linear endomorphisms of P.

In conclusion, let us turn to projective Hilbert modules of finite rank over a unital C^*-algebra A. One can show the following [63, 68].

• Let P be a right Hilbert A-module such that $\mathrm{Id}\, P \in T_A(P)$. Then P is a projective module of finite rank.

• Conversely, let P be a projective right A-module of finite rank. Then P admits a positive Hermitian form which makes it into a Hilbert module such that $\mathrm{Id}\, P \in T_A(P)$.

• Given two positive Hermitian forms $\langle .|.\rangle$ and $\langle .|.\rangle'$ on a projective right A-module P, there exists an invertible A-linear endomorphism of P such that

$$\langle p|p'\rangle' = \langle Tp|Tp'\rangle, \qquad p, p' \in P.$$

4.2 Non-commutative differential calculus

The notion of a differential calculus in Section 1.4 has been formulated for any $\mathcal{K}$-ring $\mathcal{A}$. One can generalize the Chevalley–Eilenberg differential calculus over a commutative ring in Section 1.4 to a non-commutative $\mathcal{K}$-ring $\mathcal{A}$ [32, 37, 41]. For this purpose, let us consider derivations $u \in \mathfrak{d}\mathcal{A}$ of $\mathcal{A}$. They obey the Leibniz rule

$$u(ab) = u(a)b + au(b), \qquad a, b \in \mathcal{A}, \tag{4.2.1}$$

(see Remark 1.2.1). By virtue of the relation (4.2.1), the set of derivations $\mathfrak{d}\mathcal{A}$ is both a $\mathcal{Z}_\mathcal{A}$-bimodule and a Lie $\mathcal{K}$-algebra with respect to the Lie bracket

$$[u, u'] = uu' - u'u. \tag{4.2.2}$$

It is readily observed that derivations preserve the center $\mathcal{Z}_\mathcal{A}$ of $\mathcal{A}$.

Remark 4.2.1: If $\mathcal{A}$ is an involutive ring, the differential calculus over $\mathcal{A}$ fulfills the additional relations

$$(\alpha \cdot \beta)^* = \beta^* \cdot \alpha^*, \qquad (\delta\alpha)^* = -\delta\alpha^*, \qquad \alpha, \beta \in \Omega^*. \tag{4.2.3}$$

In particular, the second relation (4.2.3) shows that δ is an antisymmetric derivation of an involutive ring $\mathcal{A}$. $\diamondsuit$

Let us consider the extended Chevalley–Eilenberg complex (1.4.4) of the Lie algebra $\mathfrak{d}\mathcal{A}$ with coefficients in the ring $\mathcal{A}$, regarded as a $\mathfrak{d}\mathcal{A}$-module. This complex contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of $\mathcal{Z}_\mathcal{A}$-multilinear skew-symmetric maps (1.4.5) with respect to the Chevalley–Eilenberg coboundary operator d (1.4.6). Its terms $\mathcal{O}^k[\mathfrak{d}\mathcal{A}]$ are $\mathcal{A}$-bimodules. The graded module $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is provided with the product (1.4.9) which obeys the relation (1.4.10) and makes $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ into a differential graded algebra. Let us note that, if $\mathcal{A}$ is not commutative, there is nothing like the graded commutativity of forms (1.4.11) in general. Since

$$\mathcal{O}^1[\mathfrak{d}\mathcal{A}] = \mathrm{Hom}_{\mathcal{Z}_\mathcal{A}}(\mathfrak{d}\mathcal{A}, \mathcal{A}), \tag{4.2.4}$$

we have the following non-commutative generalizations of the interior product

$$(u\rfloor\phi)(u_1, \ldots, u_{k-1}) = k\phi(u, u_1, \ldots, u_{k-1}), \qquad u \in \mathfrak{d}\mathcal{A}, \qquad \phi \in \mathcal{O}^*[\mathfrak{d}\mathcal{A}],$$

and the Lie derivative

$$\mathbf{L}_u(\phi) = d(u\rfloor\phi) + u\rfloor f(\phi).$$

Then one can think of elements of $\mathcal{O}^1[\mathfrak{d}\mathcal{A}]$ as being the non-commutative generalization of exterior one-forms.

The minimal Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}$ over $\mathcal{A}$ consists of the monomials

$$a_0 da_1 \wedge \cdots \wedge da_k, \qquad a_i \in \mathcal{A},$$

whose product $\wedge$ (1.4.9) obeys the juxtaposition rule

$$(a_0 da_1) \wedge (b_0 db_1) = a_0 d(a_1 b_0) \wedge db_1 - a_0 a_1 db_0 \wedge db_1, \qquad a_i, b_i \in \mathcal{A}.$$

For instance, it follows from the product (1.4.9) that, if $a, a' \in \mathcal{Z}_\mathcal{A}$, then

$$da \wedge da' = -da' \wedge da, \qquad ada' = (da')a. \tag{4.2.5}$$

PROPOSITION 4.2.1: There is the duality relation

$$\mathfrak{d}\mathcal{A} = \mathrm{Hom}_{\mathcal{A}-\mathcal{A}}(\mathcal{O}^1\mathcal{A}, \mathcal{A}), \tag{4.2.6}$$

generalizing the relation (1.3.5) to non-commutative rings. □

Outline of proof: It follows from the definition (1.4.6) of the Chevalley–Eilenberg coboundary operator that

$$(da)(u) = u(a), \qquad a \in \mathcal{A}, \qquad u \in \mathfrak{d}\mathcal{A}. \tag{4.2.7}$$

This equality yields the morphism

$$\mathfrak{d}\mathcal{A} \ni u \mapsto \phi_u \in \mathrm{Hom}_{\mathcal{A}-\mathcal{A}}(\mathcal{O}^1\mathcal{A}, \mathcal{A}), \qquad \phi_u(da) = u(a), \qquad a \in \mathcal{A}.$$

This morphism is a monomorphism because the module $\mathcal{O}^1\mathcal{A}$ is generated by elements da, $a \in \mathcal{A}$. At the same time, any element $\phi \in \mathrm{Hom}_{\mathcal{A}-\mathcal{A}}(\mathcal{O}^1\mathcal{A}, \mathcal{A})$ induces the derivation $u_\phi(a) = \phi(da)$ of $\mathcal{A}$. Thus, there is a morphism

$$\mathrm{Hom}_{\mathcal{A}-\mathcal{A}}(\mathcal{O}^1\mathcal{A}, \mathcal{A}) \to \mathfrak{d}\mathcal{A},$$

which is a monomorphism since $\mathcal{O}^1\mathcal{A}$ is generated by elements da, $a \in \mathcal{A}$. *QED*

Let us turn now to a different differential calculus over a non-commutative ring which is often used in non-commutative geometry [23, 53]. Let $\mathcal{A}$ be a (non-commutative) $\mathcal{K}$-ring over a commutative ring $\mathcal{K}$. Let us consider the tensor product $\mathcal{A} \underset{\mathcal{K}}{\otimes} \mathcal{A}$ of $\mathcal{K}$-modules. It is brought into an $\mathcal{A}$-bimodule with respect to the multiplication

$$b(a \otimes a')c = (ba) \otimes (a'c), \qquad a, a', b, c \in \mathcal{A}.$$

Let us consider its submodule $\Omega^1(\mathcal{A})$ generated by the elements

$$\mathbf{1} \otimes a - a \otimes \mathbf{1}, \qquad a \in \mathcal{A}.$$

It is readily observed that

$$d : \mathcal{A} \ni a \mapsto \mathbf{1} \otimes a - a \otimes \mathbf{1} \in \Omega^1(\mathcal{A}) \tag{4.2.8}$$

is a $\Omega^1(\mathcal{A})$-valued derivation of $\mathcal{A}$. Thus, $\Omega^1(\mathcal{A})$ is an $\mathcal{A}$-bimodule generated by the elements da, $a \in \mathcal{A}$, such that the relation

$$(da)b = d(ab) - adb, \qquad a, b \in \mathcal{A}, \tag{4.2.9}$$

holds. Let us consider the tensor algebra $\Omega^*(\mathcal{A})$ of the $\mathcal{A}$-bimodule $\Omega^1(\mathcal{A})$. It consists of the monomials

$$a_0 da_1 \cdots da_k, \qquad a_i \in \mathcal{A}, \tag{4.2.10}$$

whose product obeys the juxtaposition rule

$$(a_0 da_1)(b_0 db_1) = a_0 d(a_1 b_0) db_1 - a_0 a_1 db_0 b_1, \qquad a_i, b_i \in \mathcal{A},$$

because of the relation (4.2.9). The operator d (4.2.8) is extended to $\Omega^*(\mathcal{A})$ by the law

$$d(a_0 da_1 \cdots da_k) = da_0 da_1 \cdots da_k, \tag{4.2.11}$$

that makes $\Omega^*(\mathcal{A})$ into a differential graded algebra. Its de Rham cohomology groups are

$$H^0(\Omega^*(\mathcal{A})) = \mathcal{K}, \qquad H^{r>0}(\Omega^*(\mathcal{A})) = 0.$$

If $\mathcal{A}$ is not a unital algebra, one can consider its unital extension $\widetilde{\mathcal{A}}$ in Remark 1.1.1, and then construct the differential graded algebra $\Omega^*(\widetilde{\mathcal{A}})$. This algebra contains the differential graded subalgebra $\Omega^*(\mathcal{A})$ of monomials (4.2.10). The de Rham cohomology groups of $\Omega^*(\mathcal{A})$ are trivial.

Of course, $\Omega^*(\mathcal{A})$ is a minimal differential calculus. One calls it the *universal differential calculus* over $\mathcal{A}$ because of the following property [53]. Let P be an $\mathcal{A}$-bimodule. Any P-valued derivation Δ of $\mathcal{A}$ factorizes as $\Delta = \mathfrak{f}^\Delta \circ d$ through some $(\mathcal{A} - \mathcal{A})$-module homomorphism

$$\mathfrak{f}^\Delta : \Omega^1(\mathcal{A}) \to P. \tag{4.2.12}$$

Moreover, let $\mathcal{A}'$ be another $\mathcal{K}$-algebra and (Ω'^*, δ') its differential calculus over a $\mathcal{K}$-ring $\mathcal{A}'$. Any homomorphism $\mathcal{A} \to \mathcal{A}'$ is uniquely extended to a morphism of differential graded algebras

$$\rho^* : \Omega^*(\mathcal{A}) \to \Omega'^*$$

such that $\rho^{k+1} \circ d = \delta' \circ \rho^k$. Indeed, this morphism factorizes through the morphism of $\Omega^*(\mathcal{A})$ to the minimal differential calculus in Ω'^* which sends $da \to \delta'\rho(a)$.

Elements of the universal differential calculus $\Omega^*(\mathcal{A})$ are called *universal forms*. However, they can not be regarded as the non-commutative generalization of exterior forms because, in contrast with the Chevalley–Eilenberg differential calculus, the monomials da, $a \in \mathcal{Z}_\mathcal{A}$, of the universal differential calculus do not satisfy the relations (4.2.5). In particular, if $\mathcal{A}$ is a commutative ring, the module $\mathcal{O}^1$ (1.3.2) of exterior one-forms over

$\mathcal{A}$ is the quotient of the module $\Omega^1(\mathcal{A})$ (4.2.8) of universal forms by the relations (1.3.1). At the same time, if $P = \mathcal{A}$, the morphism (4.2.12) takes the form

$$\mathfrak{f}^\Delta(da) = \Delta(a).$$

This relation defines the monomorphism of $\Omega^1(\mathcal{A})$ to $\mathcal{O}^1[\mathfrak{d}\mathcal{A}]$ (4.2.4) by the formula (4.2.7). Therefore, its range coincides with the term $\mathcal{O}^1\mathcal{A}$ of the minimal Chevalley–Eilenberg differential calculus, i.e., there is an isomorphism

$$\Omega^1(\mathcal{A}) = \mathcal{O}^1\mathcal{A}. \tag{4.2.13}$$

4.3 Differential operators in non-commutative geometry

It seems natural to regard derivations of a non-commutative $\mathcal{K}$-ring $\mathcal{A}$ and the Chevalley–Eilenberg coboundary operator d (1.4.6) as particular differential operators in non-commutative geometry. Definition 1.2.1 provides a standard notion of differential operators on modules over a commutative ring. However, there exist its different generalizations to modules over a non-commutative ring [9, 34, 37, 56].

Let P and Q be $\mathcal{A}$-bimodules over a non-commutative $\mathcal{K}$-ring $\mathcal{A}$. The $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ of $\mathcal{K}$-linear homomorphisms $\Phi : P \to Q$ can be provided with the left $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (1.2.1) and the similar right module structures

$$(\Phi a)(p) = \Phi(p)a, \qquad (a \bullet \Phi)(p) = \Phi(pa), \quad a \in \mathcal{A}, \qquad p \in P. \tag{4.3.1}$$

For the sake of convenience, we will refer to the module structures (1.2.1) and (4.3.1) as the left and right $\mathcal{A} - \mathcal{A}^\bullet$ structures, respectively. Let us put

$$\overline{\delta}_a\Phi = \Phi a - a \bullet \Phi, \qquad a \in \mathcal{A}, \qquad \Phi \in \mathrm{Hom}_{\mathcal{K}}(P,Q). \tag{4.3.2}$$

It is readily observed that

$$\delta_a \circ \overline{\delta}_b = \overline{\delta}_b \circ \delta_a, \qquad a, b \in \mathcal{A}.$$

The left $\mathcal{A}$-module homomorphisms $\Delta : P \to Q$ obey the conditions $\delta_a\Delta = 0$, for all $a \in \mathcal{A}$ and, consequently, they can be regarded as left zero order Q-valued differential operators on P. Similarly, right zero order differential operators are defined.

Utilizing the condition (1.2.3) as a definition of a first order differential operator in non-commutative geometry, one however meets difficulties. If $P = \mathcal{A}$ and $\Delta(\mathbf{1}) = 0$, the condition (1.2.3) does not lead to the Leibniz rule (1.2.8), i.e., derivations of the $\mathcal{K}$-ring $\mathcal{A}$ are not first order differential operators. In order to overcome these difficulties, one can replace the condition (1.2.3) with the following one [34].

DEFINITION 4.3.1: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is called a first order *differential operator* of a bimodule P over a non-commutative ring $\mathcal{A}$ if it obeys the condition

$$\begin{aligned}
&\delta_a \circ \overline{\delta}_b \Delta = \overline{\delta}_b \circ \delta_a \Delta = 0, \qquad a, b \in \mathcal{A}, \\
&a\Delta(p)b - a\Delta(pb) - \Delta(ap)b + \Delta(apb) = 0, \qquad p \in P.
\end{aligned} \tag{4.3.3}$$

□

First order Q-valued differential operators on P make up a $\mathcal{Z}_{\mathcal{A}}$-module $\mathrm{Diff}_1(P, Q)$.

If P is a commutative bimodule over a commutative ring $\mathcal{A}$, then $\delta_a = \overline{\delta}_a$ and Definition 4.3.1 comes to Definition 1.2.1 for first order differential operators.

In particular, let $P = \mathcal{A}$. Any left or right zero order Q-valued differential operator Δ is uniquely defined by its value $\Delta(\mathbf{1})$. As a consequence, there are left and right $\mathcal{A}$-module isomorphisms

$$\begin{aligned}
&Q \ni q \mapsto \Delta^{\mathrm{R}}_q \in \mathrm{Diff}^{\mathrm{R}}_0(\mathcal{A}, Q), \qquad \Delta^{\mathrm{R}}_q(a) = qa, \qquad a \in \mathcal{A}, \\
&Q \ni q \mapsto \Delta^{\mathrm{L}}_q \in \mathrm{Diff}^{\mathrm{L}}_0(\mathcal{A}, Q), \qquad \Delta^{\mathrm{L}}_q(a) = aq.
\end{aligned}$$

A first order Q-valued differential operator Δ on $\mathcal{A}$ fulfils the condition

$$\Delta(ab) = \Delta(a)b + a\Delta(b) - a\Delta(\mathbf{1})b. \tag{4.3.4}$$

It is a derivation of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$. One obtains at once that any first order differential operator on $\mathcal{A}$ is split into the sums

$$\begin{aligned}
&\Delta(a) = a\Delta(\mathbf{1}) + [\Delta(a) - a\Delta(\mathbf{1})], \\
&\Delta(a) = \Delta(\mathbf{1})a + [\Delta(a) - \Delta(\mathbf{1})a]
\end{aligned}$$

of the derivations $\Delta(a) - a\Delta(\mathbf{1})$ or $\Delta(a) - \Delta(\mathbf{1})a$ and the left or right zero order differential operators $a\Delta(\mathbf{1})$ and $\Delta(\mathbf{1})a$, respectively. If u is a Q-valued derivation of $\mathcal{A}$, then au (1.2.1) and ua (4.3.1) are so for any $a \in \mathcal{Z}_{\mathcal{A}}$. Hence, Q-valued derivations of $\mathcal{A}$ constitute a $\mathcal{Z}_{\mathcal{A}}$-module $\mathfrak{d}(\mathcal{A}, Q)$. There are two $\mathcal{Z}_{\mathcal{A}}$-module decompositions

$$\begin{aligned}
&\mathrm{Diff}_1(\mathcal{A}, Q) = \mathrm{Diff}^{\mathrm{L}}_0(\mathcal{A}, Q) \oplus \mathfrak{d}(\mathcal{A}, Q), \\
&\mathrm{Diff}_1(\mathcal{A}, Q) = \mathrm{Diff}^{\mathrm{R}}_0(\mathcal{A}, Q) \oplus \mathfrak{d}(\mathcal{A}, Q).
\end{aligned}$$

They differ from each other in the inner derivations $a \mapsto aq - qa$.

Let $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{A}}(P, Q)$ and $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{A}}(P, Q)$ be the modules of right and left $\mathcal{A}$-module homomorphisms of P to Q, respectively. They are provided with the left and right $\mathcal{A} - \mathcal{A}^\bullet$-module structures (1.2.1) and (4.3.1), respectively.

PROPOSITION 4.3.2: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is a first order Q-valued differential operator on P in accordance with Definition 4.3.1 iff it obeys the condition

$$\Delta(apb) = (\overrightarrow{\partial}\, a)(p)b + a\Delta(p)b + a(\overleftarrow{\partial}\, b)(p), \qquad p \in P, \quad a, b \in \mathcal{A}, \tag{4.3.5}$$

where $\overrightarrow{\partial}$ and $\overleftarrow{\partial}$ are $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{A}}(P,Q)$- and $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{A}}(P,Q)$-valued derivations of $\mathcal{A}$, respectively [41]. Namely,

$$(\overrightarrow{\partial}\, a)(pb) = (\overrightarrow{\partial}\, a)(p)b, \qquad (\overleftarrow{\partial}\, b)(ap) = a(\overleftarrow{\partial}\, b)(p).$$

□

For instance, let P be a differential calculus over a $\mathcal{K}$-ring $\mathcal{A}$ provided with an associative multiplication $\circ$ and a coboundary operator d. Then d exemplifies a P-valued first order differential operator on P by Definition 4.3.1. It obeys the condition (4.3.5) which reads

$$d(apb) = (da \circ p)b + a(dp)b + a((-1)^{|p|}p \circ db).$$

For instance, let $P = \mathcal{O}^*\mathcal{A}$ be the Chevalley–Eilenberg differential calculus over $\mathcal{A}$. In view of the relations (4.2.4) and (4.2.6), one can think of derivations $u \in \mathfrak{d}\mathcal{A}$ as being vector fields in non-commutative geometry. A problem is that $\mathfrak{d}\mathcal{A}$ is not an $\mathcal{A}$-module. One can overcome this difficulty as follows [9].

Given a non-commutative $\mathcal{K}$-ring $\mathcal{A}$ and an $\mathcal{A}$-bimodule Q, let d be a Q-valued derivation of $\mathcal{A}$. One can think of Q as being a first degree term of a differential calculus over $\mathcal{A}$. Let Q^*_{R} be the right $\mathcal{A}$-dual of Q. It is an $\mathcal{A}$-bimodule:

$$(bu)(q) = bu(q), \qquad (ub)(q) = u(bq), \qquad b \in \mathcal{A}, \qquad q \in Q.$$

One can associate to each element $u \in Q^*_{\mathrm{R}}$ the $\mathcal{K}$-module morphism

$$\widehat{u} : \mathcal{A} \ni a \mapsto u(da) \in \mathcal{A}. \tag{4.3.6}$$

This morphism obeys the relations

$$\widehat{(bu)}(a) = bu(da), \qquad \widehat{u}(ba) = \widehat{u}(b)a + \widehat{(ub)}(a). \tag{4.3.7}$$

One calls $(Q^*_{\mathrm{R}}, u \mapsto \widehat{u})$ the $\mathcal{A}$-right *Cartan pair*, and regards $\widehat{u}$ (4.3.6) as an $\mathcal{A}$-valued first order differential operator on $\mathcal{A}$ [9]. Let us note that $\widehat{u}$ (4.3.6) need not be a derivation of $\mathcal{A}$ and fails to satisfy Definition 4.3.1, unless u belongs to the two-sided $\mathcal{A}$-dual $Q^* \subset Q^*_{\mathrm{R}}$ of Q. Morphisms $\widehat{u}$ (4.3.6) are called into play in order to describe (left) vector fields in non-commutative geometry [9, 48].

In particular, if $Q = \mathcal{O}^1\mathcal{A}$, then au for any $u \in \mathfrak{d}\mathcal{A}$ and $a \in \mathcal{A}$ is a left non-commutative vector field in accordance with the relation (1.4.7).

Similarly, the $\mathcal{A}$-left Cartan pair is defined. For instance, ua for any $u \in \mathfrak{d}\mathcal{A}$ and $a \in \mathcal{A}$ is a right *non-commutative vector field*.

If $\mathcal{A}$-valued derivations $u_1, \ldots u_r$ of a non-commutative $\mathcal{K}$-ring $\mathcal{A}$ or the above mentioned non-commutative vector fields $\widehat{u}_1, \ldots \widehat{u}_r$ on $\mathcal{A}$ are regarded as first order differential operators on $\mathcal{A}$, it seems natural to think of their compositions $u_1 \circ \cdots u_r$ or $\widehat{u}_1 \circ \cdots \widehat{u}_r$

as being particular higher order differential operators on $\mathcal{A}$. Let us turn to the general notion of a differential operator on $\mathcal{A}$-bimodules.

By analogy with Definition 1.2.1, one may try to generalize Definition 4.3.1 by means of the maps δ_a (1.2.2) and $\overline{\delta}_a$ (4.3.2). A problem lies in the fact that, if $P = Q = \mathcal{A}$, the compositions $\delta_a \circ \delta_b$ and $\overline{\delta}_a \circ \overline{\delta}_b$ do not imply the Leibniz rule and, as a consequence, compositions of derivations of $\mathcal{A}$ fail to be differential operators [41, 77].

This problem can be solved if P and Q are regarded as left $\mathcal{A}$-modules [56]. Let us consider the $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ provided with the left $\mathcal{A} - \mathcal{A}^\bullet$ module structure (1.2.1). We denote by $\mathcal{Z}_0$ its center, i.e., $\delta_a \Phi = 0$ for all $\Phi \in \mathcal{Z}_0$ and $a \in \mathcal{A}$. Let $\mathcal{I}_0 = \overline{\mathcal{Z}}_0$ be the $\mathcal{A} - \mathcal{A}^\bullet$ submodule of $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ generated by $\mathcal{Z}_0$. Let us consider:

(i) the quotient $\mathrm{Hom}_{\mathcal{K}}(P, Q)/\mathcal{I}_0$,

(ii) its center $\mathcal{Z}_1$,

(iii) the $\mathcal{A} - \mathcal{A}^\bullet$ submodule $\overline{\mathcal{Z}}_1$ of $\mathrm{Hom}_{\mathcal{K}}(P, Q)/\mathcal{I}_0$ generated by $\mathcal{Z}_1$,

(iv) the $\mathcal{A} - \mathcal{A}^\bullet$ submodule $\mathcal{I}_1$ of $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ given by the relation $\mathcal{I}_1/\mathcal{I}_0 = \overline{\mathcal{Z}}_1$.

Then we define the $\mathcal{A} - \mathcal{A}^\bullet$ submodules $\mathcal{I}_r$, $r = 2, \ldots$, of $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ by induction as $\mathcal{I}_r/\mathcal{I}_{r-1} = \overline{\mathcal{Z}}_r$, where $\overline{\mathcal{Z}}_r$ is the $\mathcal{A} - \mathcal{A}^\bullet$ module generated by the center $\mathcal{Z}_r$ of the quotient $\mathrm{Hom}_{\mathcal{K}}(P, Q)/\mathcal{I}_{r-1}$.

Definition 4.3.3: Elements of the submodule $\mathcal{I}_r$ of $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ are said to be left r-order Q-valued *differential operators* of an $\mathcal{A}$-bimodule P [56]. □

Proposition 4.3.4: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is a differential operator of order r in accordance with Definition 4.3.3 iff it is a finite sum

$$\Delta(p) = b_i \Phi^i(p) + \Delta_{r-1}(p), \qquad b_i \in \mathcal{A}, \tag{4.3.8}$$

where Δ_{r-1} and $\delta_a \Phi^i$ for all $a \in \mathcal{A}$ are $(r-1)$-order differential operators if $r > 0$, and they vanish if $r = 0$ [41]. □

If $\mathcal{A}$ is a commutative ring, Definition 4.3.3 comes to Definition 1.2.1. Indeed, the expression (4.3.8) shows that $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is an r-order differential operator iff $\delta_a \Delta$ for all $a \in \mathcal{A}$ is a differential operator of order $r - 1$.

Proposition 4.3.5: If P and Q are $\mathcal{A}$-bimodules, the set $\mathcal{I}_r$ of r-order Q-valued differential operators on P is provided with the left and right $\mathcal{A} - \mathcal{A}^\bullet$ module structures [41]. □

Let $P = Q = \mathcal{A}$. Any zero order differential operator on $\mathcal{A}$ in accordance with Definition 4.3.3 takes the form $a \mapsto cac'$ for some $c, c' \in \mathcal{A}$.

Proposition 4.3.6: Let Δ_1 and Δ_2 be n- and m-order $\mathcal{A}$-valued differential operators on $\mathcal{A}$, respectively. Then their composition $\Delta_1 \circ \Delta_2$ is an $(n+m)$-order differential operator [41]. □

Any derivation $u \in \mathfrak{d}\mathcal{A}$ of a $\mathcal{K}$-ring $\mathcal{A}$ is a first order differential operator in accordance with Definition 4.3.3. Indeed, it is readily observed that

$$(\delta_a u)(b) = au(b) - u(ab) = -u(a)b, \qquad b \in \mathcal{A},$$

is a zero order differential operator for all $a \in \mathcal{A}$. The compositions au, $u \bullet a$ (1.2.1), ua, $a \bullet u$ (4.3.1) for any $u \in \mathfrak{d}\mathcal{A}$, $a \in \mathcal{A}$ and the compositions of derivations $u_1 \circ \cdots \circ u_r$ are also differential operators on $\mathcal{A}$ in accordance with Definition 4.3.3.

At the same time, non-commutative vector fields do not satisfy Definition 4.3.3 in general. First order differential operators by Definition 4.3.1 also need not obey Definition 4.3.3, unless $P = Q = \mathcal{A}$.

By analogy with Definition 4.3.3 and Proposition 4.3.4, one can define differential operators on right $\mathcal{A}$-modules as follows.

DEFINITION 4.3.7: Let P and Q be seen as right $\mathcal{A}$-modules over a non-commutative $\mathcal{K}$-ring $\mathcal{A}$. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is said to be a right zero order Q-valued differential operator on P if it is a finite sum $\Delta = \Phi^i b_i$, $b_i \in \mathcal{A}$, where $\overline{\delta}_a \Phi^i = 0$ for all $a \in \mathcal{A}$. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is called a right differential operator of order $r > 0$ on P if it is a finite sum

$$\Delta(p) = \Phi^i(p) b_i + \Delta_{r-1}(p), \qquad b_i \in \mathcal{A}, \tag{4.3.9}$$

where Δ_{r-1} and $\overline{\delta}_a \Phi^i$ for all $a \in \mathcal{A}$ are right $(r-1)$-order differential operators. $\Box$

Definition 4.3.3 and Definition 4.3.7 of left and right differential operators on $\mathcal{A}$-bimodules are not equivalent, but one can combine them as follows.

DEFINITION 4.3.8: Let P and Q be bimodules over a non-commutative $\mathcal{K}$-ring $\mathcal{A}$. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is a two-sided zero order Q-valued differential operator on P if it is either a left or right zero order differential operator. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is said to be a two-sided differential operator of order $r > 0$ on P if it is brought both into the form

$$\Delta = b_i \Phi^i + \Delta_{r-1}, \qquad b_i \in \mathcal{A},$$

and

$$\Delta = \overline{\Phi}^i \overline{b}_i + \overline{\Delta}_{r-1}, \qquad \overline{b}_i \in \mathcal{A},$$

where Δ_{r-1}, $\overline{\Delta}_{r-1}$ and $\delta_a \Phi^i$, $\overline{\delta}_a \overline{\Phi}^i$ for all $a \in \mathcal{A}$ are two-sided $(r-1)$-order differential operators. $\Box$

One can think of this definition as a generalization of Definition 4.3.1 to higher order differential operators.

It is readily observed that two-sided differential operators described by Definition 4.3.8 need not be left or right differential operators, and *vice versa*. At the same time, $\mathcal{A}$-valued derivations of a $\mathcal{K}$-ring $\mathcal{A}$ and their compositions obey Definition 4.3.8.

4.4 Connections in non-commutative geometry

This Section is devoted to the definitions of a connection in non-commutative geometry. We follow the notion of an algebraic connection in Section 1.3 generalized to modules over non-commutative rings [22, 23, 35, 36, 37].

Let (Ω^*, δ) be a differential calculus over a $\mathcal{K}$-ring $\mathcal{A}$, and let P be a left $\mathcal{A}$-module. Following Definition 1.3.2, one can construct the tensor product of modules $\Omega^1 \otimes P$ and then define a *left connection* on P as a $\mathcal{K}$-module morphism

$$\nabla^L : P \to \Omega^1 \underset{\mathcal{A}}{\otimes} P, \tag{4.4.1}$$

which obeys the Leibniz rule

$$\nabla^L(ap) = \delta a \otimes p + a\nabla^L(p), \qquad p \in P, \qquad a \in \mathcal{A},$$

[53, 87]. If $\Omega^* = \Omega^*(\mathcal{A})$ is the universal differential calculus over $\mathcal{A}$, the connection (4.4.1) is called a *universal connection*.

Similarly, a *right non-commutative connection* on a right $\mathcal{A}$-module P is defined as a $\mathcal{K}$-module morphism

$$\nabla^R : P \to P \underset{\mathcal{A}}{\otimes} \Omega^1,$$

which obeys the Leibniz rule

$$\nabla^R(pa) = p \otimes \delta a + \nabla^R(p)a, \qquad p \in P, \qquad a \in \mathcal{A}.$$

The forthcoming theorem shows that a connection on a left (or right) module over a non-commutative ring need not exist [27, 53].

THEOREM 4.4.1: A left (resp. right) universal connection on a left (resp. right) module P of finite rank exists iff P is projective. □

In contrast with connections on left and right modules, there is a problem how to define a connection on an $\mathcal{A}$-bimodule over a non-commutative ring. If $\mathcal{A}$ is a commutative ring, then the tensor products $\Omega^1 \otimes P$ and $P \otimes \Omega^1$ are naturally identified by the permutation morphism

$$\varrho : \alpha \otimes p \mapsto p \otimes \alpha, \qquad \alpha \in \Omega^1, \quad p \in P,$$

so that any left connection ∇^L is also the right one $\varrho \circ \nabla^L$, and *vice versa*. If $\mathcal{A}$ is a non-commutative ring, neither left nor right connections are connections on an $\mathcal{A}$-bimodule P since $\nabla^L(P) \in \Omega^1 \otimes P$, whereas $\nabla^R(P) \in P \otimes \Omega^1$. As a palliative, one may assume that there exists some $\mathcal{A}$-bimodule isomorphism

$$\varrho : \Omega^1 \otimes P \to P \otimes \Omega^1. \tag{4.4.2}$$

Then a couple (∇^L, ∇^R) of right and left non-commutative connections on P is said to be a ϱ-compatible if $\varrho\circ\nabla^L = \nabla^R$ [36, 53, 64] (see also [28] for a weaker condition). However, this couple is not a true connection on an $\mathcal{A}$-bimodule. The problem is not simplified even if $P = \Omega^1$, together with the natural permutation

$$\phi\otimes\phi' \mapsto \phi'\otimes\phi, \qquad \phi, \phi' \in \Omega^1.$$

A different construction of a connection on $\mathcal{A}$-bimodules has been suggested in [35]. Let $(\Omega^*\mathcal{A}, \delta)$ be the minimal Chevalley–Eilenberg differential calculus over a $\mathcal{K}$-ring $\mathcal{A}$. Due to the isomorphism (4.2.6), any element $u\in\mathfrak{d}\mathcal{A}$ determines the morphisms

$$u : \mathcal{O}^1\mathcal{A}\otimes P \ni \alpha\otimes p \mapsto u(\alpha)p \in P, \tag{4.4.3}$$
$$u : P\otimes\mathcal{O}^1\mathcal{A} \ni \pi\otimes\alpha \mapsto pu(\alpha) \in P. \tag{4.4.4}$$

DEFINITION 4.4.2: A connection on an $\mathcal{A}$-bimodule P with respect to the minimal Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}$ over $\mathcal{A}$ is defined as a $\mathcal{Z}_\mathcal{A}$-bimodule morphism

$$\nabla : \mathfrak{d}\mathcal{A} \ni u \mapsto \nabla_u \in \mathrm{Hom}_\mathcal{K}(P, P), \tag{4.4.5}$$

which obeys the Leibniz rule

$$\nabla_u(a_i p b_j) = u(a_i)pb_j + a_i\nabla_u(p)b_j + a_i p u(b_j), \quad p\in P, \quad a_k, b_k \in A_k. \tag{4.4.6}$$

□

Comparing the formulae (4.3.5) and (4.4.6) shows that, by virtue of Proposition 4.3.2, ∇_u (4.4.6) is a first order differential operator on P in accordance with Definitions 4.3.1 and 4.3.7.

Let us recall that, if $a\in\mathcal{Z}_\mathcal{A}$, then δa belongs to the center $\mathcal{Z}_P$ of the module P and $u(\delta a)\in\mathcal{Z}_\mathcal{A}$ for all $u\in\mathfrak{d}\mathcal{A}$. If $\mathcal{A}$ is a commutative ring, Definition 4.4.2 is identic to Definition 1.3.3.

Remark 4.4.1: Due to the isomorphism (4.2.13), Definition 4.4.2 can be also extended to connections with respect to the universal differential calculus. ♢

Remark 4.4.2: Definition 4.4.2 can be applied to the case of a non-minimal differential calculus Ω^* over $\mathcal{A}$ by replacing the derivation module $\mathfrak{d}\mathcal{A}$ with the dual $(\Omega^1)^*$ of the $\mathcal{A}$-bimodule Ω^1. Definition 4.4.2 is straightforwardly applied to the Chevalley–Eilenberg differential calculus $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ over $\mathcal{A}$ due to the inclusion

$$\mathfrak{d}\mathcal{A}\subset\mathfrak{d}\mathcal{A}^{**} = (\mathcal{O}^*[\mathfrak{d}\mathcal{A}])^*.$$

♢

We agree to call a connection in Definition 4.4.2 the *Dubois–Violette connection*. It is readily observed that any left (resp. right) connection ∇ on an $\mathcal{A}$-module of type $(1,0)$

(resp. $(0,1)$) determines the Dubois–Violette connection $\nabla_u = u\circ\nabla$, $u\in\mathfrak{d}\mathcal{A}$, on P due to the morphism (4.4.3) (resp. (4.4.4)).

Example 4.4.3: If $P=\mathcal{A}$, the morphisms

$$\nabla_u(a) = u(a), \quad u\in\mathfrak{d}\mathcal{A}, \quad a\in\mathcal{A}, \tag{4.4.7}$$

define the canonical Dubois–Violette connection on the ring $\mathcal{A}$. Then, due to the Leibniz rule (4.4.6), a Dubois–Violette connection on any $\mathcal{A}$-bimodule P is also a Dubois–Violette connection on P, seen as a $\mathcal{Z}_\mathcal{A}$-bimodule. $\diamondsuit$

Example 4.4.4: If P is an $\mathcal{A}$-bimodule and the ring $\mathcal{A}$ has only inner derivations

$$\mathrm{ad}\, b(a) = ba - ab,$$

the morphisms

$$\nabla_{\mathrm{ad}\, b}(p) = bp - pb, \quad b\in\mathcal{A}, \quad p\in P, \tag{4.4.8}$$

define the canonical Dubois–Violette connection on P. $\diamondsuit$

The curvature R of a Dubois–Violette connection ∇ (4.4.5) on an $\mathcal{A}$-module P is defined as the $\mathcal{Z}_\mathcal{A}$-module morphism

$$\begin{aligned} &R: \mathfrak{d}\mathcal{A}\times\mathfrak{d}\mathcal{A}\ni (u,u')\mapsto R_{u,u'}\in \mathrm{Hom}\,_{A_i-A_j}(P,P), \\ &R_{u,u'}(p) = \nabla_u(\nabla_{u'}(p)) - \nabla_{u'}(\nabla_u(p)) - \nabla_{[u,u']}(p), \quad p\in P, \end{aligned} \tag{4.4.9}$$

[35]. We have the relations

$$\begin{aligned} &R_{au,a'u'} = aa'R_{u,u'}, \qquad a,a'\in\mathcal{Z}_\mathcal{A}, \\ &R_{u,u'}(a_i p b_j) = a_i R_{u,u'}(p) b_j, \quad a_k, b_k\in A_k. \end{aligned}$$

For instance, the curvature of the connections (4.4.7) and (4.4.8) vanishes.

There are the following standard constructions of new Dubois–Violette connections from old ones.

(i) Given $\mathcal{A}$-bimodules P and P' and two Dubois–Violette connections ∇ and ∇' on them, there is an obvious Dubois–Violette connection $\nabla\oplus\nabla'$ on the direct sum $P\oplus P'$.

(ii) Let P be an $\mathcal{A}$-bimodule and P^* its $\mathcal{A}$-dual. For any Dubois–Violette connection ∇ on P, there is a unique *dual Dubois–Violette connection* ∇' on P^* such that

$$u(\langle p,p'\rangle) = \langle \nabla_u(p), p'\rangle + \langle p, \nabla'(p')\rangle, \quad p\in P, \quad p'\in P^*, \quad u\in\mathfrak{d}\mathcal{A}.$$

(iii) Let P and P' be $\mathcal{A}$-bimodules, and let ∇ and ∇' be Dubois–Violette connections on these modules. For any $u\in\mathfrak{d}\mathcal{A}$, let us consider the endomorphism

$$(\nabla\otimes\nabla')_u = \nabla_u\otimes \mathrm{Id}\, P' + \mathrm{Id}\, P\otimes\nabla'_u \tag{4.4.10}$$

of the tensor product $P \otimes P'$ of $\mathcal{K}$-modules P and P'. This endomorphism preserves the subset of $P \otimes P'$ generated by elements

$$pa \otimes p' - p \otimes ap', \qquad p \in P, \quad p' \in P' \quad a \in A_k.$$

Consequently, the endomorphisms (4.4.10) define a Dubois–Violette connection on the tensor product $P \otimes P'$ of modules P and P'.

(iv) Let $\mathcal{A}$ be a unital $*$-algebra and P an involutive module over $\mathcal{A}$. Let us recall that, in this case, the derivation module $\mathfrak{d}\mathcal{A}$ consists of only symmetric derivations of $\mathcal{A}$. For any Dubois–Violette connection ∇ on P, the conjugate Dubois–Violette connection ∇^* on P is defined by the relation

$$\nabla^*_u(p) = (\nabla_u(p^*))^*. \tag{4.4.11}$$

A Dubois–Violette connection ∇ on an involutive module P is said to be *real* if $\nabla = \nabla^*$.

Let now $P = \mathcal{O}^1\mathcal{A}$. Any Dubois–Violette connection on an $\mathcal{A}$-bimodule $\mathcal{O}^1\mathcal{A}$ is called a *linear connection* [35]. Let us note that this is not the term for a left or right connection on $\mathcal{O}^1\mathcal{A}$ [36]. If $\mathcal{O}^1\mathcal{A}$ is an involutive module, a linear connection on it is assumed to be real. Given a linear connection ∇ on $\mathcal{O}^1\mathcal{A}$, there is an $\mathcal{A}$-module homomorphism

$$\begin{aligned} &T : \mathcal{O}^1\mathcal{A} \to \mathcal{O}^2\mathcal{A}, \\ &(T\phi)(u,u') = (d\phi)(u,u') - \nabla_u(\phi)(u') + \nabla_{u'}(\phi)(u), \end{aligned} \tag{4.4.12}$$

called the *torsion* of the linear connection ∇.

4.5 Matrix geometry

Matrix geometry over the algebra $\mathcal{A} = M_n$ of complex $n\times n$ matrices provides an important example of a non-commutative system of finite degrees of freedom [33, 58]. Let $\{\varepsilon_r\}$, $1 \leq r \leq n^2-1$, be an anti-Hermitian basis for the (right) Lie algebra $su(n)$. All derivations of the algebra M_n are inner, and $u_r = \operatorname{ad}\varepsilon_r$ constitute a basis for the complex Lie algebra $\mathfrak{d}M_n$ of derivations of M_n, together with the commutation relations

$$[u_r, u_q] = c^s_{rq} u_s,$$

where c^s_{rq} are structure constants of the Lie algebra $su(n)$. Since the center $\mathcal{Z}_{M_n}$ of M_n consists of matrices $c\mathbf{1}$, $c \in \mathbb{C}$, the derivation module $\mathfrak{d}M_n$ is an (n^2-1)-dimensional complex vector space. Let us consider the minimal Chevalley–Eilenberg differential calculus $(\mathcal{O}^*M_n, d)$ over the algebra M_n with respect to the Chevalley–Eilenberg coboundary operator d (1.4.6). In particular, $\mathcal{O}^0M_n = M_n$, while $\mathcal{O}^1M_n$ is a free left M_n-module of rank n^2-1 whose basis $\{\theta^r\}$ is the dual of the basis $\{u_r\}$ for the complex Lie algebra $\mathfrak{d}M_n$, i.e.,

$$\theta^r(u_q) = \delta^r_q \mathbf{1}.$$

It is readily observed that elements θ^r of this basis belong to the center of the M_n-bimodule $\mathcal{O}^1 M_n$, i.e.,

$$a\theta^r = \theta^r a, \qquad a \in M_n. \tag{4.5.1}$$

It also follows that

$$\theta^r \wedge \theta^q = -\theta^q \wedge \theta^r. \tag{4.5.2}$$

The morphism

$$d : M_n \to \mathcal{O}^1 M_n,$$

given by the formula (1.4.7), reads

$$d\varepsilon_r(u_q) = \operatorname{ad}\varepsilon_q(\varepsilon_r) = c^s_{qr}\varepsilon_s,$$

that is,

$$d\varepsilon_r = c^s_{qr}\varepsilon_s\theta^q. \tag{4.5.3}$$

The formula (1.4.8) leads to the Maurer–Cartan equations

$$d\theta^r = -\frac{1}{2}c^r_{qs}\theta^q \wedge \theta^s. \tag{4.5.4}$$

If we put $\theta = \varepsilon_r\theta^r$, the equality (4.5.3) can be brought into the form

$$da = a\theta - \theta a, \qquad a \in M_n.$$

It follows that the M_n-bimodule $\mathcal{O}^1 M_n$ is generated by only one element θ.

Let us consider linear connections on the M_n-bimodule $\mathcal{O}^1 M_n$ in matrix geometry. Such a connection ∇ is given by the relations

$$\begin{aligned} &\nabla_{u=c^r u_r} = c^r\nabla_r, \\ &\nabla_r(\theta^p) = \omega^p_{rq}\theta^q, \qquad \omega^p_{rq} \in M_n. \end{aligned} \tag{4.5.5}$$

Bearing in mind the equalities (4.5.1) – (4.5.2), we obtain from the Leibniz rule (4.4.6) that

$$a\nabla_r(\theta^p) = \nabla_r(\theta^p)a, \qquad a \in M_n.$$

It follows that elements ω^p_{rq} in the expression (4.5.5) are proportional to the identity $\mathbf{1} \in M_n$, i.e., are complex numbers. Then the relations

$$\nabla_r(\theta^p) = \omega^p_{rq}\theta^q, \qquad \omega^p_{rq} \in \mathbb{C}, \tag{4.5.6}$$

determine any linear connection on $\mathcal{O}^1 M_n$. Let us point out the following two particular linear connections on $\mathcal{O}^1 M_n$.

(i) Since all derivations of the algebra M_n are inner, we have the curvature-free connection (4.4.8) given by the relations $\nabla_r(\theta^p) = 0$. However, this connection is not torsion-free. The expressions (4.5.4) and (4.4.12) result in

$$(T\theta^p)(u_r, u_q) = -c^p_{rq}.$$

(ii) In matrix geometry, there is a unique torsion-free linear connection

$$\nabla_r(\theta^p) = -c^p_{rq}\theta^q.$$

4.6 Connes' non-commutative geometry

Connes' non-commutative geometry addresses the representation of the universal differential calculus $(\Omega^*(\mathcal{A}), d)$ over an involutive algebra $\mathcal{A}$ in a Hilbert space E so that the coboundary operator da, $a \in \mathcal{A}$, is represented by the bracket $[\mathcal{D}, \pi(a)]$, $a \in A$, where $\mathcal{D}$ is a certain operator in E. Thus, one comes to the notion of a spectral triple [23, 53, 87].

DEFINITION 4.6.1: A *spectral triple* $(\mathcal{A}, E, \mathcal{D})$ is given by a unital $*$-algebra $\mathcal{A} \subset B(E)$ of bounded operators in a separable Hilbert space E and a (unbounded) self-adjoint operator $\mathcal{D}$ in E such that:

(i) the brackets $[\mathcal{D}, a]$, $a \in A$, are bounded operators in E,

(ii) the resolvent $(\mathcal{D} - \lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is a compact operator in E, $\Box$

The triple $(\mathcal{A}, E, \mathcal{D})$ is also called the *K-cycle* over $\mathcal{A}$ [23].

Remark 4.6.1: In Connes' commutative geometry (see Example 4.6.3 below), $\mathcal{A}$ is the algebra $\mathbb{C}^\infty(X)$ of smooth complex functions on a compact manifold X. It is a dense subalgebra of the C^*-algebra of continuous complex functions on X. Generalizing this case, one usually assume that an algebra $\mathcal{A}$ in Connes' spectral triple is a dense involutive subalgebra of some C^*-algebra. $\diamondsuit$

Remark 4.6.2: In many cases, E is a $\mathbb{Z}_2$-graded Hilbert space whose grading automorphism γ obeys the conditions

$$\gamma\mathcal{D} + \mathcal{D}\gamma = 0, \qquad [a, \gamma] = 0, \qquad a \in \mathcal{A},$$

i.e., elements of $\mathcal{A}$ are even operators, while $\mathcal{D}$ is the odd one. The spectral triple is called *even* if such a grading exists and *odd* otherwise. $\diamondsuit$

Given a spectral triple $(\mathcal{A}, E, \mathcal{D})$ in Definition 4.6.1, let $(\Omega^*(\mathcal{A}), d)$ be the universal differential calculus over the algebra $\mathcal{A}$. Let us consider the representation of the graded algebra $\Omega^*(\mathcal{A})$ by bounded operators

$$\pi(a_0 da_1 \cdots da_k) = a_0[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k] \qquad (4.6.1)$$

in the Hilbert space E. Since

$$[\mathcal{D}, a]^* = -[\mathcal{D}, a^*],$$

we have

$$\pi(\omega)^* = \pi(\omega^*), \qquad \omega \in \Omega^*(\mathcal{A}).$$

However, the representation (4.6.1) fails to be a representation of the differential graded algebra $\Omega^*(\mathcal{A})$ because $\pi(\omega) = 0$ need not imply that $\pi(d\omega) = 0$. Therefore, one should construct the appropriate quotient of $\Omega^*(\mathcal{A})$ in order to obtain a differential graded algebra of operators in E.

Let J_0^* be the graded two-sided ideal of $\Omega^*(\mathcal{A})$ where

$$J_0^k = \{\omega \in \Omega^k(\mathcal{A}) \,:\, \pi(\omega) = 0\}.$$

Then it is readily observed that

$$J^* = J_0^* + dJ_0^*$$

is a differential graded two-sided ideal of $\Omega^*(\mathcal{A})$. By *Connes' differential calculus* is meant the pair $(\Omega^*_{\mathcal{D}}\mathcal{A}, d)$ such that

$$\Omega^*_{\mathcal{D}}\mathcal{A} = \Omega^*\mathcal{A}/J^*, \qquad d[\omega] = [d\omega],$$

where $[\omega]$ denotes the class of $\omega \in \Omega^*(\mathcal{A})$ in $\Omega^*_{\mathcal{D}}\mathcal{A}$. It is a differential calculus over $\Omega^0_{\mathcal{D}}\mathcal{A} = \mathcal{A}$. Its representation $\pi(\Omega^*_{\mathcal{D}}\mathcal{A})$ is given by the classes $\pi[\omega]$ of operators

$$\sum_j a_0^j[\mathcal{D}, a_1^j] \cdots [\mathcal{D}, a_k^j], \qquad a_i^j \in \mathcal{A},$$

modulo the operators

$$\sum_j [\mathcal{D}, b_0^j][\mathcal{D}, b_1^j] \cdots [\mathcal{D}, b_{k-1}^j]$$

such that

$$\sum_j b_0^j[\mathcal{D}, b_1^j] \cdots [\mathcal{D}, b_{k-1}^j] = 0\}.$$

It should be emphasized that $\pi([\omega])$ are not operators in E. The coboundary operator in $\pi(\Omega^*_{\mathcal{D}}\mathcal{A})$ reads

$$d(a_0[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k]) = [\mathcal{D}, a_0][\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k].$$

Example 4.6.3: Connes' commutative geometry is characterized by the spectral triple $(\mathcal{A}, E, \mathcal{D})$ where:

• $\mathcal{A} = \mathbb{C}^\infty(X)$ is the above mentioned algebra of smooth complex functions on a compact manifold X,

• $E = L^2(X, S)$ is the Hilbert space of square integrable sections of a spinor bundle $S \to X$,

• $\mathcal{D}$ is the Dirac operator on this spinor bundle.

In this case, the representation π (4.6.1) of the universal differential calculus $\Omega^*(\mathcal{A})$ over $\mathcal{A}$ (up to a possible twisting by a complex line bundle) is the complex Clifford algebra $\mathrm{Cliff}(T^*X) \otimes \mathbb{C}$ of the cotangent bundle T^*X and

$$\pi(\Omega^*_{\mathcal{D}}\mathcal{A}) = \mathcal{O}^*(X) \otimes \mathbb{C}$$

is the algebra of complex exterior forms on X [24, 38, 66]. $\diamondsuit$

Spectral triples have been studied, e.g., for non-commutative tori, the Moyal deformations of $\mathbb{R}^n$, non-commutative spheres 2-, 3- and 4-spheres [16, 26], and quantum Heisenberg manifolds [17].

Given Connes' differential calculus $(\Omega^*_{\mathcal{D}}\mathcal{A}, d)$ over an algebra $\mathcal{A}$, let P be a right projective $\mathcal{A}$-module of finite rank. In the spirit of the Serre–Swan theorem 1.6.3, one can think of P as being a non-commutative vector bundle. By virtue of Theorem 4.4.1, it admits a connection. Let us construct this connection in an explicit form.

Given a generic right finite projective module P over a complex ring $\mathcal{A}$, let

$$\mathbf{p} : \mathbb{C}^N \underset{\mathbb{C}}{\otimes} \mathcal{A} \to P, \qquad i_P : P \to \mathbb{C}^N \underset{\mathbb{C}}{\otimes} \mathcal{A}$$

be the corresponding projection and injection, where $\otimes_{\mathbb{C}}$ denotes the tensor product over $\mathbb{C}$, while $\otimes_{\mathcal{A}}$ stands for the tensor product over the ring $\mathcal{A}$. There is the composition of morphisms

$$P \xrightarrow{i_p} \mathbb{C}^N \underset{\mathbb{C}}{\otimes} \mathcal{A} \xrightarrow{\mathrm{Id}\otimes d} \mathbb{C}^N \underset{\mathbb{C}}{\otimes} \Omega^1(\mathcal{A}) \xrightarrow{\mathbf{p}} P \underset{\mathcal{A}}{\otimes} \Omega^1(\mathcal{A}), \tag{4.6.2}$$

where the canonical module isomorphism

$$\mathbb{C}^N \underset{\mathbb{C}}{\otimes} \Omega^1(\mathcal{A}) = (\mathbb{C}^N \underset{\mathbb{C}}{\otimes} \mathcal{A}) \underset{\mathcal{A}}{\otimes} \Omega^1(\mathcal{A})$$

is used. It is readily observed that the composition (4.6.2) denoted briefly as $\mathbf{p} \circ d$ is a right universal connection on the module P.

Given the universal connection $\mathbf{p} \circ d$ on a right finite projective module P over a $*$-algebra $\mathcal{A}$, let us consider the morphism

$$P \xrightarrow{\mathbf{p}\circ d} P \underset{\mathcal{A}}{\otimes} \Omega^1(\mathcal{A}) \xrightarrow{\mathrm{Id}\otimes\pi} P \underset{\mathcal{A}}{\otimes} \Omega^1_{\mathcal{D}}\mathcal{A}.$$

This is a right connection ∇_0 on the module P with respect to Connes' differential calculus. Any other right connection ∇ on P with respect to Connes' differential calculus takes the form

$$\nabla = \nabla_0 + \sigma = (\mathrm{Id} \otimes \pi) \circ \mathbf{p} \circ d + \sigma \tag{4.6.3}$$

where σ is an $\mathcal{A}$ module morphism

$$\sigma : P \to P \underset{\mathcal{A}}{\otimes} \Omega^1_{\mathcal{D}}\mathcal{A}.$$

The term σ in the connection ∇ (4.6.3) is called a *non-commutative gauge field.*

4.7 Differential calculus over Hopf algebras

Hopf algebras make a contribution to many quantum models [18, 19, 59]. For instance, quantum groups are particular Hopf algebras. In a general setting, any non-cocommutative Hopf algebra can be treated as a quantum group [18]. However, the development of differential calculus and differential geometry over Hopf algebras has met some problems [41].

Let $\mathcal{A}$ be a complex unital algebra, i.e., a $\mathbb{C}$-ring. The *tensor product* $\mathcal{A}\otimes\mathcal{A}$ of algebras $\mathcal{A}$ is defined as that of vector spaces $\mathcal{A}$ provided with the multiplication

$$(a \otimes b) \otimes (a' \otimes b') = (aa') \otimes (bb').$$

Let us write the multiplication operation of the algebra $\mathcal{A}$ as a $\mathbb{C}$-linear morphism

$$m : a \otimes b \to ab, \qquad a, b \in \mathcal{A}.$$

A *coalgebra* $\mathcal{A}$ is defined as a vector space $\mathcal{A}$ provided with the following linear morphisms:

- a coassociative *comultiplication* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$,
- a *counit* $\epsilon : \mathcal{A} \to \mathbb{C}$,

which obey the relations

$$(\Delta \otimes \mathrm{Id}\,) \circ \Delta = (\mathrm{Id}\, \otimes \Delta) \circ \Delta, \quad (\epsilon \otimes \mathrm{Id}\,) \circ \Delta = (\mathrm{Id}\, \otimes \epsilon) \circ \Delta = \mathrm{Id}\,. \tag{4.7.1}$$

As a shorthand, one writes

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}, \quad (\Delta \otimes \mathrm{Id}\,) \circ \Delta(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}. \tag{4.7.2}$$

A comultiplication and a counit extend to tensor products pairwise, i.e.,

$$\Delta(a \otimes a') = \Delta(a) \otimes \Delta(a'), \qquad \epsilon(a \otimes a') = \epsilon(a)\epsilon(a').$$

A *bi-algebra* $(\mathcal{A}, m, \Delta, \epsilon)$ is defined as an associative algebra $\mathcal{A}$ which is also a coalgebra so that

$$\begin{aligned}
&\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \\
&\sum (ab)_{(1)} \otimes (ab)_{(2)} = \sum (a_{(1)}b_{(1)}) \otimes (a_{(2)}b_{(2)}), \qquad a, b \in \mathcal{A}, \\
&\epsilon(\mathbf{1}) = 1, \qquad \epsilon(ab) = \epsilon(a)\epsilon(b).
\end{aligned}$$

A *Hopf algebra* $(\mathcal{A}, m, \Delta, \epsilon, S)$ is a bi-algebra $\mathcal{A}$ endowed with a linear morphism $S : \mathcal{A} \to \mathcal{A}$, called the *antipode*, such that

$$m((S \otimes \mathrm{Id}\,)\Delta(a)) = m((\mathrm{Id}\, \otimes S)\Delta(a)) = \epsilon(a)\mathbf{1}.$$

It obeys the relations

$$S(\mathbf{1}) = \mathbf{1}, \qquad S(ab) = S(b)S(a), \quad a, b \in \mathcal{A},$$
$$\epsilon \circ S = \epsilon, \qquad \Delta \circ S = P \circ (S \otimes S) \circ \Delta,$$

where $P : a \otimes b \mapsto b \otimes a$ is the transposition operator. Let us note that, given a bi-algebra $\mathcal{A}$, there is a unique antipode, if any, such that $\mathcal{A}$ becomes a Hopf algebra.

For the sake of brevity, we call Δ, ϵ and S the *co-operations* of a Hopf algebra.

A Hopf algebra is said to be *cocommutative* if $P \circ \Delta = \Delta$. If a Hopf algebra is commutative or cocommutative, then $S^2 = \mathrm{Id}\,$. If $(\mathcal{A}, m, \Delta, \epsilon, S)$ is a Hopf algebra whose antipode S is invertible, then

$$\mathcal{A}^t = (\mathcal{A}, m, P \circ \Delta, \epsilon, S^{-1}) \tag{4.7.3}$$

is also a Hopf algebra.

Let $\mathcal{A}$ be an involutive algebra. A Hopf algebra $(\mathcal{A}, m, \Delta, \epsilon, S)$ is said to be *involutive* if

$$\Delta(a^*) = \Delta(a)^* = \sum a^*_{(1)} \otimes a^*_{(2)},$$
$$\epsilon(a^*) = \overline{\epsilon(a)}, \qquad S(S(a^*)^*) = a.$$

A Hopf algebra is called *quasi-triangular* if there exists an invertible element $\mathcal{R} = r_i \otimes r^i \in \mathcal{A} \otimes \mathcal{A}$ which obeys the relations

$$(\Delta \otimes \mathrm{Id}\,)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{4.7.4a}$$
$$(\mathrm{Id}\, \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{12}\mathcal{R}_{23}, \tag{4.7.4b}$$
$$(P \circ \Delta)(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \qquad a \in \mathcal{A}, \tag{4.7.4c}$$

where

$$\mathcal{R}_{12} = r_k \otimes r^k \otimes \mathbf{1}, \qquad \mathcal{R}_{13} = r_k \otimes \mathbf{1} \otimes r^k, \qquad \mathcal{R}_{23} = \mathbf{1} \otimes r_k \otimes r^k.$$

The element $\mathcal{R}$ is said to be the *universal R-matrix* of a Hopf algebra $\mathcal{A}$. It satisfies the *quantum Yang–Baxter equation*

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{4.7.5}$$

One can show that

$$(S \otimes \mathrm{Id}\,)(\mathcal{R}) = \mathcal{R}^{-1}, \qquad (\mathrm{Id}\, \otimes S)(\mathcal{R}^{-1}) = \mathcal{R}, \qquad (S \otimes S)(\mathcal{R}) = \mathcal{R}.$$

The antipode of a quasi-triangular Hopf algebra is always invertible.

Example 4.7.1: Let $\mathfrak{g}$ be a finite-dimensional Lie algebra whose basis is $\{e_k\}$. Its universal enveloping algebra $\overline{\mathfrak{g}}$ is provided with the structure of a cocommutative Hopf algebra $U\mathfrak{g}$, called the *classical Hopf algebra*, with respect to the co-operations

$$\Delta(e_k) = e_k \otimes \mathbf{1} + \mathbf{1} \otimes e_k, \qquad \epsilon(e_k) = 0, \qquad S(e_k) = -e_k,$$

extended by linearity to $\overline{\mathfrak{g}}$. It is a quasi-triangular Hopf algebra where $\mathcal{R} = \mathbf{1} \otimes \mathbf{1}$. $\diamondsuit$

Example 4.7.2: Let G be a finite group and $\mathbb{C}G$ the complex group ring [41]. It is brought into the cocommutative Hopf algebra, called the *group Hopf algebra*, characterized by the co-operations

$$\Delta(g) = g \otimes g, \qquad \epsilon(g) = 1, \qquad S(g) = g^{-1}, \qquad g \in G,$$

extended by linearity to $\mathbb{C}G$. $\diamondsuit$

Example 4.7.3: Given a finite group G, let us consider the algebra $\mathbb{C}(G) = \mathbb{C}^G$ of complex functions on G. Let us identify

$$\mathbb{C}(G) \otimes \mathbb{C}(G) = \mathbb{C}(G \times G).$$

Then the algebra $\mathbb{C}(G)$ is a commutative Hopf algebra with respect to the co-operations

$$\Delta(f)(g, g') = f(gg'), \qquad \epsilon(f) = f(\mathbf{1}), \qquad S(f)(g) = f(g^{-1}) \tag{4.7.6}$$

for all $f \in \mathbb{C}(G)$. It is called the *group function Hopf algebra*. $\diamondsuit$

Example 4.7.4: The Hopf algebra $U_q(b_+)$, where q is a non-zero real number, is generated by the elements $\mathbf{1}$, a, g and g^{-1} obeying the relations

$$\begin{aligned}
& gg^{-1} = g^{-1}g = \mathbf{1}, \qquad ga = qag, \\
& \Delta(a) = a \otimes \mathbf{1} + g \otimes a, \qquad \Delta(g) = g \otimes g, \qquad \Delta(g^{-1}) = g^{-1} \otimes g^{-1}, \\
& \epsilon(a) = 0, \qquad \epsilon(g) = \epsilon(g^{-1}) = 1, \\
& S(a) = -g^{-1}a, \qquad S(g) = g^{-1}, \qquad S(g^{-1}) = g.
\end{aligned}$$

$\diamondsuit$

Example 4.7.5: Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra. It yields an associated complex Lie group G of $n \times n$ complex matrices b which obey a family of polynomial equations $p(b) = 0$. Correspondingly, we have an algebraic variety with the coordinate algebra $\mathbb{C}[G]$ of polynomials $\mathbb{C}[b^i_j]$ in n^2 variables modulo the relations $p(b) = 0$. It is a commutative Hopf algebra with respect to the co-operations

$$\Delta(b^i_j) = b^i_k \otimes b^k_j, \qquad \epsilon(b^i_j) = \delta^i_j.$$

Its antipode is given algebraically via a matrix of cofactors of the matrix b^i_j, i.e.,

$$S(b^i_k)b^k_j = b^i_k S(b^k_j) = \delta^i_j \mathbf{1}.$$

For instance, the Hopf algebra $\mathbb{C}[SL(2)]$ is generated by four elements a, b, c and d which are assembled into the matrix

$$A = \begin{pmatrix} a\, b \\ c\, d \end{pmatrix}$$

and obey the polynomial equation

$$A \det A = (\det A)A = A,$$

i.e., $\det A = ad - cb$ is the unit element. The co-operations of this Hopf algebra can be written in the compact form

$$\Delta(A) = A \otimes A, \qquad \epsilon(A) = \begin{pmatrix} 1\, 0 \\ 0\, 1 \end{pmatrix}, \qquad S(A) = A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

It is quasi-triangular Hopf algebra where $\mathcal{R} = \mathbf{1} \otimes \mathbf{1}$. $\diamondsuit$

Given a Hopf algebra $(\mathcal{A}, m, \Delta, \epsilon, S)$, let us consider the space $\mathcal{A}^* = \mathrm{Hom}\,(A, \mathbb{C})$ of complex linear forms on $\mathcal{A}$. It is a complex ring with respect to the *convolution product*

$$\begin{aligned} &f * f' = (f \otimes f') \circ \Delta, \qquad f, f' \in \mathcal{A}^*, \\ &(f * f')(a) = \sum f(a_{(1)}) f'(a_{(2)}), \qquad a \in A. \end{aligned} \tag{4.7.7}$$

The unit of $\mathcal{A}^*$ is the counit ϵ of $\mathcal{A}$. Sometimes, the notion of the convolution product (4.7.7) includes the linear morphisms

$$\begin{aligned} &f * a = [(\mathrm{Id} \otimes f) \circ \Delta](a) = \sum a_{(1)} f(a_{(2)}), \\ &a * f = [(f \otimes \mathrm{Id}) \circ \Delta](a) = \sum f(a_{(1)}) a_{(2)}, \quad a \in \mathcal{A}, \end{aligned} \tag{4.7.8}$$

of the complex space $\mathcal{A}$ [88]. There are the relations

$$(f * f')(a) = f'(a * f) = f(f' * a), \qquad a \in \mathcal{A}, \qquad f, f' \in \mathcal{A}^*. \tag{4.7.9}$$

A Hopf algebra $(\mathcal{A}', m', \Delta', \epsilon', S')$ is said to be *dually paired* with a Hopf algebra $(\mathcal{A}, m, \Delta, \epsilon, S)$ if there exists a non-degenerate interior product

$$\langle ., . \rangle : \mathcal{A} \otimes \mathcal{A}' \to \mathbb{C}$$

which satisfies the relations

$$\langle a, a'b' \rangle = \langle \Delta(a), a' \otimes b' \rangle, \qquad a, b \in \mathcal{A}, \quad a', b' \in \mathcal{A}', \tag{4.7.10a}$$

$$\langle a \otimes b, \Delta'(a') \rangle = \langle ab, a' \rangle, \tag{4.7.10b}$$

$$\epsilon'(a') = \langle \mathbf{1}, a' \rangle, \qquad \epsilon(a) = \langle a, \mathbf{1} \rangle, \tag{4.7.10c}$$

$$\langle a, S'(a') \rangle = \langle S(a), a' \rangle, \tag{4.7.10d}$$

where $\langle , \rangle$ extends to tensor products pairwise, i.e.,

$$\langle a \otimes b, a' \otimes b' \rangle = \langle a, a' \rangle \langle b, b' \rangle. \tag{4.7.11}$$

If $\mathcal{A}$ is an involutive Hopf algebra, the involutive Hopf algebra $\mathcal{A}'$ dually paired with $\mathcal{A}$ should satisfy the additional condition

$$\langle a^*, a' \rangle = \langle a, (S'(a'))^* \rangle.$$

For instance, the Hopf algebra $U_q(b_+)$ in Example 4.7.4 is dually paired with itself by

$$\langle a, a \rangle = 1, \qquad \langle g, g \rangle = q, \qquad \langle a, g \rangle = \langle g, a \rangle = 0.$$

If $\mathfrak{g}$ is a finite-dimensional complex semi-simple Lie algebra represented by $n \times n$ matrices $\rho(a)$, $a \in \mathfrak{g}$, the Hopf algebras $U\mathfrak{g}$ in Example 4.7.1 and $\mathbb{C}[G]$ in Example 4.7.5 are dually paired with respect to the interior product

$$\langle a, b^i_j \rangle = \rho(a)^i_j. \tag{4.7.12}$$

Let a Hopf algebra $(\mathcal{A}, m, \Delta, \epsilon, S)$ be a finite-dimensional vector space, and let $\mathcal{A}'$ be its algebraic dual. Then the equality (4.7.11) yields an isomorphism

$$(\mathcal{A} \otimes \mathcal{A})' = \mathcal{A}' \otimes \mathcal{A}'.$$

In this case, the relations (4.7.10a) – (4.7.10d) provide $\mathcal{A}'$ with a unique Hopf algebra structure $(m', \Delta', \epsilon', S')$, called the *dual Hopf algebra*.

One says that a Hopf algebra $\mathcal{H}$ acts on a complex algebra A on the left (or A is a *module over a Hopf algebra* $\mathcal{H}$) if $\mathcal{H}$ acts $h \triangleright a$ on A as a vector space such that

$$h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \qquad a, b \in A,$$
$$h \triangleright \mathbf{1} = \epsilon(h)\mathbf{1}, \qquad h \in \mathcal{H},$$

i.e., the multiplication and the unit in A commute with the action of $\mathcal{H}$.

In particular, any Hopf algebra acts on itself as an algebra. Namely, the left and right *adjoint action* of a Hopf algebra $\mathcal{H}$ on itself is given by the formulae

$$h \triangleright h' : h \otimes h' \mapsto \sum h_{(1)} h' S(h_{(2)}), \tag{4.7.13}$$

$$h' \triangleleft h : h \otimes h' \mapsto \sum S(h_{(1)}) h' h_{(2)}. \tag{4.7.14}$$

For instance, the left and right adjoint action of the classical Hopf algebra $U\mathfrak{g}$ in Example 4.7.1 reads

$$a \triangleright b = [a, b], \qquad b \triangleleft a = [b, a], \qquad a, b \in \mathfrak{g}. \tag{4.7.15}$$

The left adjoint action of the group Hopf algebra $\mathbb{C}G$ in Example 4.7.2 coincides with the adjoint action of the group G on itself. Since the Hopf algebras $\mathbb{C}(G)$ in Example 4.7.3 and $\mathbb{C}[G]$ in Example 4.7.5 are commutative, their adjoint action is trivial.

If a Hopf algebra $\mathcal{H}'$ is dually paired with a Hopf algebra $\mathcal{H}$, its action on $\mathcal{H}$ reads

$$h' \triangleright h = \sum h_{(1)}\langle h', h_{(2)}\rangle, \qquad h' \in \mathcal{H}', \qquad h \in \mathcal{H}. \tag{4.7.16}$$

For instance, the Hopf algebra $\mathbb{C}[G]$ acts on the classical Hopf algebra $U\mathfrak{g}$ by the law

$$b^i_j \triangleright a = \mathbf{1}\rho(a)^i_j + a\delta^i_j.$$

A coalgebra $(\mathcal{A}, \Delta, \epsilon)$ is said to be a module of a Hopf algebra $\mathcal{H}$ if it is an $\mathcal{H}$-module as an algebra and, additionally, the relations

$$\Delta(h \triangleright a) = \sum (h_{(1)} \triangleright a_{(1)}) \otimes (h_{(2)} \triangleright a_{(2)}), \qquad \epsilon(h \triangleright a) = \epsilon(h)\epsilon(a)$$

hold for all $h \in \mathcal{H}$ and $a \in \mathcal{A}$.

A left *coaction* of a Hopf algebra $\mathcal{H}$ on a vector space V is given by a map $\beta : V \to \mathcal{H} \otimes V$ such that

$$(\mathrm{Id} \otimes \beta) \circ \beta = (\Delta \otimes \mathrm{Id}) \circ \beta, \qquad (\epsilon \otimes \mathrm{Id}) \circ \beta = \mathrm{Id}. \tag{4.7.17}$$

A right coaction $V \to V \otimes \mathcal{H}$ of $\mathcal{H}$ on V is similarly defined. We write

$$\beta(v) = \sum \overline{v}^{(1)} \otimes v^{(2)}, \qquad \overline{v}^{(1)} \in \mathcal{H}, \qquad v^{(2)} \in V. \tag{4.7.18}$$

One says that V is a (left) *comodule* over a Hopf algebra $\mathcal{H}$. For instance the *trivial comodule* is $\mathbb{C}$ where

$$\beta(\lambda) = \mathbf{1} \otimes \lambda, \qquad \lambda \in \mathbb{C}.$$

If V is a $\mathcal{H}$-bimodule, the additional relations

$$\beta(hvh') = \Delta(h)\beta(v)\Delta(h'), \qquad h, h' \in \mathcal{H}, \tag{4.7.19}$$

hold. If an element $v \in V$ satisfies the equality

$$\beta(v) = \mathbf{1} \otimes v, \tag{4.7.20}$$

one says that v is left-invariant.

A Hopf algebra $\mathcal{H}$ coacts on a unital algebra A if A is a $\mathcal{H}$-comodule and β is an algebra homomorphism of A to the algebra $\mathcal{H} \otimes A$, i.e.,

$$\beta(ab) = \beta(a)\beta(b), \qquad \beta(\mathbf{1}_A) = \mathbf{1} \otimes \mathbf{1}_A, \qquad a, b \in A.$$

Then A is called a (left) $\mathcal{H}$*-comodule algebra*. A glance at the formulae (4.7.1) and (4.7.17) shows that any Hopf algebra coacts

$$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$$

on itself as an algebra.

A Hopf algebra coacts $\mathcal{H}$ on a Hopf algebra $\mathcal{A}$ if it coacts on $\mathcal{A}$ as an algebra and, additionally, this coaction commutates with co-operations of $\mathcal{A}$. In particular, any Hopf algebra coacts on itself on the left as a coalgebra by the law

$$\mathrm{ad}_l(h) = \sum h_{(1)}S(h_{(3)}) \otimes h_{(2)} \tag{4.7.21}$$

(see the shorthand (4.7.2)). Accordingly, the right coaction reads

$$\mathrm{ad}_r(h) = \sum h_{(2)} \otimes S(h_{(1)})h_{(3)}. \tag{4.7.22}$$

Turn now to the differential calculus over Hopf algebras.

Let $(\mathcal{A}, m, \Delta, \epsilon, S)$ be a Hopf algebra with an invertible antipode S. A differential calculus over $\mathcal{A}$ is that over $\mathcal{A}$ as a non-commutative ring which obeys some additional conditions related to the co-operations in $\mathcal{A}$ [49, 89].

Here, we focus on a minimal first order differential calculus (henceforth *FODC*) over $\mathcal{A}$ (see Remark 4.7.6 below). It is a left $\mathcal{A}$-module Ω^1 generated by elements da, $a \in \mathcal{A}$, and provided with the $\mathcal{A}$-bimodule structure by the rule

$$(da)b = d(ab) - adb, \qquad a, b \in \mathcal{A}. \tag{4.7.23}$$

The universal FODC $\Omega^1(\mathcal{A})$ over $\mathcal{A}$ is isomorphic to the sub-bimodule $\mathrm{Ker}\, m$ of the $\mathcal{A}$-bimodule $\mathcal{A}\otimes\mathcal{A}$. Then any FODC Ω^1 over $\mathcal{A}$ is uniquely characterized by a sub-bimodule

$$\mathcal{N} = \{\sum(a_k \otimes b_k - a_k b_k \otimes \mathbf{1}) \in \mathrm{Ker}\, m \,:\, \Omega^1 \ni \sum a_k db_k = 0\} \tag{4.7.24}$$

of the $\mathcal{A}$-bimodule $\mathrm{Ker}\, m$. Obiously, $\Omega^1 = \Omega^1(\mathcal{A})/\mathcal{N}$.

A FODC (Ω^1, d) over a Hopf algebra $\mathcal{A}$ is called *left-covariant* if it possesses the structure of a left $\mathcal{A}$-comodule

$$\Delta_l : \Omega^1 \to \mathcal{A} \otimes \Omega^1$$

(see relations (4.7.17) and (4.7.19)) such that

$$\Delta_l(adb) = \Delta(a)(\mathrm{Id} \otimes d)\Delta(b), \qquad a, b \in \mathcal{A}. \tag{4.7.25}$$

Let us note that the original notion of a left-covariant FODC in [89] implies the above mentioned one. Similarly, a FODC over a Hopf algebra $\mathcal{A}$ is called *right-covariant* if it possesses the structure of a right $\mathcal{A}$-comodule $\Delta_r : \Omega^1 \to \Omega^1 \otimes \mathcal{A}$ such that

$$\Delta_r(adb) = \Delta(a)(d \otimes \mathrm{Id}\,)\Delta(b), \qquad a, b \in \mathcal{A}.$$

A FODC over $\mathcal{A}$ is called *bicovariant* if it is both a left- and right-covariant FODC satisfying the relation

$$(\mathrm{Id} \otimes \Delta_r) \circ \Delta_l = (\Delta_l \otimes \mathrm{Id}\,) \circ \Delta_r.$$

We here restrict our consideration to left-covariant FODCs over a Hopf algebra $\mathcal{A}$. An $\mathcal{A}$-bimodule V which is also an $\mathcal{A}$-comodule is called a *covariant bimodule*. Covariant bimodules possess the following important properties.

THEOREM 4.7.1: Let (V,Δ_l) be a left-covariant bimodule over a Hopf algebra $\mathcal{A}$, and let $V_{\rm inv}$ denote the complex vector space of left-invariant elements of V obeying the condition (4.7.20). There exists an epimorphism of complex spaces $\rho : V \to V_{\rm inv}$ such that

$$\rho(av) = \epsilon(a)\rho(v), \qquad v \in V, \qquad a \in \mathcal{A}. \tag{4.7.26}$$

Moreover, if

$$\Delta_l(v) = \sum a_k \otimes v_k, \qquad a_k \in \mathcal{A}, \qquad v_k \in V, \tag{4.7.27}$$

then

$$\rho(v) = \sum S(a_k)v_k. \tag{4.7.28}$$

□

THEOREM 4.7.2: Let $\{w_i\}_{i\in I}$ be a basis for the vector space $V_{\rm inv}$ of left-invariant elements of V in Theorem 4.7.1. Then the following hold.

- Any element of V is uniquely decomposed into the finite sums

$$v = \sum a_i w_i, \qquad v = \sum w_i b_i, \qquad a_i, b_i \in \mathcal{A}. \tag{4.7.29}$$

- There exist complex linear forms $f_{ij} \in \mathcal{A}^*$ on the complex vector space $\mathcal{A}$ such that

$$w_i b = \sum_j (f_{ij} * b) w_j, \qquad b \in \mathcal{A}, \tag{4.7.30}$$

$$a w_i = \sum_j w_j (f_{ij} \circ S^{-1} * a), \qquad a \in \mathcal{A}, \tag{4.7.31}$$

where $*$ is the convolution product (4.7.7) – (4.7.8).

- The forms f_{ij} are uniquely defined. They obey the relations

$$f_{ij}(ab) = \sum_k f_{ik}(a) f_{kj}(b), \qquad f_{ij}(\mathbf{1}) = \delta_{ij}, \qquad a, b \in \mathcal{A}. \tag{4.7.32}$$

□

Theorem 4.7.2 describes all left-covariant bimodules. Namely, given a family of complex linear forms $\{f_{ij}\}_{i,j\in I}$ obeying the conditions (4.7.30) – (4.7.31), one can consider a free left $\mathcal{A}$-module V generated by elements v_i indexed by the set I and provided it with the operations

$$w_i b = \sum_j (f_{ij} * b) w_j, \qquad \Delta_l(a w_i) = \Delta(a)(\mathrm{Id} \otimes w_i). \tag{4.7.33}$$

For instance, if $V = \Omega^1$ is a left-covariant FODC over $\mathcal{A}$, the condition

$$\Delta_l(db) = \sum b_{(1)} \otimes db_{(2)}, \qquad b \in \mathcal{A},$$

(4.7.25) is exactly the relation (4.7.27) in Theorem 4.7.1. Therefore, the projection ρ of an element $db \in \Omega^1$ to the space $V_{\rm inv}$ is

$$\rho(db) = \sum S(b_{(1)})db_{(2)} \tag{4.7.34}$$

in accordance with the formula (4.7.26). It is called the *Maurer–Cartan form*. By virtue of the equality (4.7.27), the projection ρ of an arbitrary element $\phi \in \Omega^1$ reads

$$\rho(\phi) = \rho(\sum_k a_k db_k) = \sum_k \epsilon(a_k) \sum S(b_{k(1)})db_{k(2)}.$$

It follows that, for any element $\phi \in \Omega^1$, there exists an element $b \in \operatorname{Ker}\epsilon$ such that $\rho(\phi) = \rho(db)$. Namely, we have

$$\rho(\sum_k a_k db_k) = \rho(db),$$
$$b = \sum_k \epsilon(a_k)(b_k - \epsilon(b_k)\mathbf{1}).$$

Consequently, any invariant element of a left-covariant FODC Ω^1 is of the form (4.7.34) where $b \in \operatorname{Ker}\epsilon$.

The covariance conditions impose some restrictions on the sub-bimodule $\mathcal{N}$ (4.7.24). The well-known Woronowicz theorem states the following [89].

THEOREM 4.7.3: Let $\mathcal{R}$ be a right ideal of $\mathcal{A}$ contained in $\operatorname{Ker}\epsilon$ and

$$\mathcal{N} = \{\sum aS(b_{(1)}) \otimes b_{(2)} \,:\, b \in \mathcal{R},\, a \in \mathcal{A}\}. \tag{4.7.35}$$

Then $\mathcal{N}$ is a sub-bimodule of the universal FODC $\Omega^1(\mathcal{A})$, and

$$\Omega^1_{\mathcal{N}} = \Omega^1(\mathcal{A})/\mathcal{N} \tag{4.7.36}$$

is a left-covariant FODC. Moreover, any left-covariant FODC can be obtained in this way. □

It follows that

$$\Omega^1_{\mathcal{N}} \ni da = 0$$

iff either $a = \mathbf{1}$ of $a \in \mathcal{R}$. In particular, the universal FODC over a Hopf algebra is always left-covariant, right-covariant and bicovariant.

Given the right ideal $\mathcal{R}$ in Theorem 4.7.3, let us consider the subspace

$$T = \{\chi \in \mathcal{A}^* \,:\, \chi(\mathbf{1}) = 0;\, \chi(a) = 0,\, a \in \mathcal{R}\} \tag{4.7.37}$$

of the complex dual $\mathcal{A}^*$ of $\mathcal{A}$. Somebody calls T the *quantum tangent space* [46]. Moreover, T is a complex Lie algebra with respect to the bracket

$$[\chi, \chi'] = \chi * \chi' - \chi' * \chi.$$

Therefore, it is also called the *quantum Lie algebra* [81]. One can show the following.

• There is the interior product

$$\begin{aligned}
&\langle .,.\rangle : \Omega^1_{\mathcal{N}} \times T \to \mathbb{C}, \\
&\langle \phi, \chi\rangle = \chi(a), \qquad \rho(da) = \rho(\phi), \qquad a \in \operatorname{Ker}\epsilon,
\end{aligned} \tag{4.7.38}$$

which obeys the relations

$$\langle a\phi, \chi\rangle = \epsilon(a)\langle \phi.\chi\rangle, \qquad \langle da, \chi\rangle = \chi(a).$$

• The interior product (4.7.38) makes the space $V_{\rm inv}$ of left-invariant elements of the FODC $\Omega^1_{\mathcal{N}}$ (4.7.36) and the quantum tangent space T (4.7.37) into a dual pair.

• In order to simplify the notation, let us assume that T is finite-dimensional. Let $\{\chi_i\}_{i\in I}$ be a basis for T, $\{w_i\}$ the dual basis for $V_{\rm inv}$ and $\{a_j\}$ the set of elements of $\operatorname{Ker}\epsilon$ defined by the condition $\chi_i(a_j) = \delta_{ij}$. Then the relations

$$\begin{aligned}
&w_i = \rho(da_i), \qquad f_{ij}(b) = \chi_i(a_j b), \qquad b \in \mathcal{A}, \\
&\phi = \sum_i \langle \phi, \chi_i\rangle w_i, \qquad da = \sum_i \chi_i(a) w_i, \qquad a \in \mathcal{A}, \\
&\chi_i * ab = \sum_j (\chi_i * a)(f_{ij} * b) + a(\chi_i * b)
\end{aligned} \tag{4.7.39}$$

hold (see Theorem 4.7.2 for the notation).

In view of the relation (4.7.39), one can think of the elements

$$u \in \operatorname{Hom}(\mathcal{A}, \mathcal{A}), \qquad u(a) = \chi * a = \sum a_{(1)}\chi(a_{(2)}) \tag{4.7.40}$$

as being vector fields for the left-covariant FODC $\Omega^1_{\mathcal{N}}$ which obey the deformed Leibniz rule (4.7.39). They are called *invariant vector fields* [3], and can be defined in an intrinsic way as follows.

PROPOSITION 4.7.4: An element $u \in \operatorname{Hom}(\mathcal{A}, \mathcal{A})$ is an invariant vector field iff it satisfies the equality

$$u = (\mathrm{Id} \otimes \epsilon) \circ (\mathrm{Id} \otimes u) \circ \Delta.$$

□

However, the notion of invariant vector fields meets the standard problem of non-commutative geometry investigated in Section 4.3. Namely, the space of invariant vector fields fails to be an $\mathcal{A}$-module. For instance, one has suggested to overcome this difficulty

by appealing to Cartan pairs in Section 4.3. The general problem of differential operators on Hopf algebras has been studied in [56].

Remark 4.7.6: Let us say a few words on a higher order differential calculus over a Hopf algebra. A minimal differential calculus (Ω^*, d) over a Hopf algebra $\mathcal{A}$ seen as an associative algebra is said to be left-covariant if Ω^* possesses the structure of a left $\mathcal{A}$-comodule

$$\Delta_l : \Omega^* \to \mathcal{A} \otimes \Omega^*$$

such that

$$\Delta_l(a_0 da_1 \cdots da_k) = \Delta(a_0)[(\mathrm{Id} \otimes d)\Delta(a_1)] \cdots [(\mathrm{Id} \otimes d)\Delta(a_k)], \quad a_i \in \mathcal{A}.$$

Let $\Omega^*(\mathcal{A})$ be the universal differential calculus over $\mathcal{A}$ and $\mathcal{N}^*$ its differential graded ideal such that

$$\Delta_l(\mathcal{N}^*) \subset \mathcal{A} \otimes \mathcal{N}^*.$$

Then, the differential calculus $\Omega^*(\mathcal{A})/\mathcal{N}^*$ is left-covariant. Similarly, right-covariant differential calculi over a Hopf algebra $\mathcal{A}$ are considered. Let us note that different ideals $\mathcal{N}^*$ may lead to isomorphic quotients $\Omega^*(\mathcal{A})/\mathcal{N}^*$. We always choose $\mathcal{N}^*$ the kernel of the morphism $\Omega^*(\mathcal{A}) \to \Omega^*$. $\diamondsuit$

Chapter 5

Appendix. Cohomology

For the sake of convenience of the reader, several topics on cohomology are compiled in this Chapter.

5.1 Cohomology of complexes

This Section summarizes the relevant basics on homology and cohomology of complexes of modules over a commutative ring [57, 62].

Let $\mathcal{K}$ be a commutative ring. A sequence

$$0 \to B^0 \xrightarrow{\delta^0} B^1 \xrightarrow{\delta^1} \cdots B^p \xrightarrow{\delta^p} \cdots \tag{5.1.1}$$

of modules B^p and their homomorphisms δ^p is said to be a *cochain complex* (henceforth, simply, a *complex*) if

$$\delta^{p+1} \circ \delta^p = 0, \qquad p \in \mathbb{N},$$

i.e., $\operatorname{Im} \delta^p \subset \operatorname{Ker} \delta^{p+1}$. The homomorphisms δ^p are called *coboundary operators*. For the sake of convenience, let us denote $B^{-1} = 0$ and $\delta^{-1} : 0 \to B^0$. Elements of the module B^p are said to be *p-cochains*, while elements of its submodules $\operatorname{Ker} \delta^p \subset B^p$ and $\operatorname{Im} \delta^{p-1} \subseteq \operatorname{Ker} \delta^p$ are called *p-cocycles* and *p-coboundaries*, respectively. The *p-th cohomology group* of the complex B^* (5.1.1) is the factor module

$$H^p(B^*) = \operatorname{Ker} \delta^p / \operatorname{Im} \delta^{p-1}.$$

It is a $\mathcal{K}$-module. In particular, $H^0(B^*) = \operatorname{Ker} \delta^0$.

A complex (5.1.1) is said to be *exact* at a term B^p if $H^p(B^*) = 0$. It is an exact sequence if all cohomology groups are trivial.

A complex (B^*, δ^*) is called *acyclic* if its cohomology groups $H^{p>0}(B^*)$ are trivial. It is acyclic if there exists a *homotopy operator* $\mathbf{h}$, defined as a set of module morphisms

$$\mathbf{h}^{p+1} : B^{p+1} \to B^p, \qquad p \in \mathbb{N},$$

such that

$$\mathbf{h}^{p+1} \circ \delta^p + \delta^{p-1} \circ \mathbf{h}^p = \mathrm{Id}\, B^p, \qquad p \in \mathbb{N}_+.$$

Indeed, if $\delta^p b^p = 0$, then $b^p = \delta^{p-1}(\mathbf{h}^p b^p)$, and $H^{p>0}(B^*) = 0$.

A complex (B^*, δ^*) is said to be a *resolution* of a module B if it is acyclic and $H^0(B^*) = B$.

The following are the standard constructions of new complexes from old ones.

• Given complexes (B_1^*, δ_1^*) and (B_2^*, δ_2^*), their *direct sum* $B_1^* \oplus B_2^*$ is a complex of modules

$$(B_1^* \oplus B_2^*)^p = B_1^p \oplus B_2^p$$

with respect to the coboundary operators

$$\delta_\oplus^p (b_1^p + b_2^p) = \delta_1^p b_1^p + \delta_2^p b_2^p.$$

• Given a subcomplex (C^*, δ^*) of a complex (B^*, δ^*), the *factor complex* B^*/C^* is defined as a complex of factor modules B^p/C^p provided with the coboundary operators

$$\delta^p [b^p] = [\delta^p b^p],$$

where $[b^p] \in B^p/C^p$ denotes the coset of the element b^p.

• Given complexes (B_1^*, δ_1^*) and (B_2^*, δ_2^*), their *tensor product* $B_1^* \otimes B_2^*$ is a complex of modules

$$(B_1^* \otimes B_2^*)^p = \mathop{\oplus}_{k+r=p} B_1^k \otimes B_2^r$$

with respect to the coboundary operators

$$\delta_\otimes^p (B_1^k \otimes B_2^r) = (\delta_1^k b_1^k) \otimes b_2^r + (-1)^k b_1^k \otimes (\delta_2^r b_2^r).$$

A *cochain morphism* of complexes

$$\gamma : B_1^* \to B_2^* \tag{5.1.2}$$

is defined as a family of degree-preserving homomorphisms

$$\gamma^p : B_1^p \to B_2^p, \qquad p \in \mathbb{N},$$

which commute with the coboundary operators, i.e.,

$$\delta_2^p \circ \gamma^p = \gamma^{p+1} \circ \delta_1^p, \qquad p \in \mathbb{N}.$$

It follows that if $b^p \in B_1^p$ is a cocycle or a coboundary, then $\gamma^p(b^p) \in B_2^p$ is so. Therefore, the cochain morphism of complexes (5.1.2) yields an induced homomorphism of their cohomology groups

$$[\gamma]^* : H^*(B_1^*) \to H^*(B_2^*). \tag{5.1.3}$$

Let us consider a short sequence of complexes

$$0 \to C^* \xrightarrow{\gamma} B^* \xrightarrow{\zeta} F^* \to 0, \tag{5.1.4}$$

represented by the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C^p \arrow[r, "\delta_C^p"] \arrow[d, "\gamma_p"'] & C^{p+1} \arrow[r] \arrow[d, "\gamma_{p+1}"'] & \cdots \\
\cdots \arrow[r] & B^p \arrow[r, "\delta_B^p"] \arrow[d, "\zeta_p"'] & B^{p+1} \arrow[r] \arrow[d, "\zeta_{p+1}"'] & \cdots \\
\cdots \arrow[r] & F^p \arrow[r, "\delta_F^p"] \arrow[d] & F^{p+1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

It is said to be exact if all columns of this diagram are exact, i.e., γ is a cochain monomorphism and ζ is a cochain epimorphism onto the quotient $F^* = B^*/C^*$.

THEOREM 5.1.1: The short exact sequence of complexes (5.1.4) yields the long exact sequence of their cohomology groups

$$\begin{aligned} 0 \to H^0(C^*) &\xrightarrow{[\gamma]^0} H^0(B^*) \xrightarrow{[\zeta]^0} H^0(F^*) \xrightarrow{\tau^0} H^1(C^*) \longrightarrow \cdots \\ &\longrightarrow H^p(C^*) \xrightarrow{[\gamma]^p} H^p(B^*) \xrightarrow{[\zeta]^p} H^p(F^*) \xrightarrow{\tau^p} H^{p+1}(C^*) \longrightarrow \cdots . \end{aligned} \tag{5.1.5}$$

□

THEOREM 5.1.2: A direct sequence of complexes

$$B_0^* \longrightarrow B_1^* \longrightarrow \cdots B_k^* \xrightarrow{\gamma_{k+1}^k} B_{k+1}^* \longrightarrow \cdots \tag{5.1.6}$$

admits a direct limit B_∞^* which is a complex whose cohomology $H^*(B_\infty^*)$ is a direct limit of the direct sequence of cohomology groups

$$H^*(B_0^*) \longrightarrow H^*(B_1^*) \longrightarrow \cdots H^*(B_k^*) \xrightarrow{[\gamma_{k+1}^k]} H^*(B_{k+1}^*) \longrightarrow \cdots .$$

This statement is also true for a direct system of complexes indexed by an arbitrary directed set. □

5.2 Cohomology of Lie algebras

One can associate to an arbitrary Lie algebra the Chevalley–Eilenberg complex. In this Section, $\mathfrak{g}$ denotes a Lie algebra (not necessarily finite-dimensional) over a commutative ring $\mathcal{K}$.

Let $\mathfrak{g}$ act on a $\mathcal{K}$-module P on the left by endomorphisms

$$\mathfrak{g}\times P\ni(\varepsilon,p)\mapsto\varepsilon p\in P,$$
$$[\varepsilon,\varepsilon']p=(\varepsilon\circ\varepsilon'-\varepsilon'\circ\varepsilon)p,\qquad \varepsilon,\varepsilon'\in\mathfrak{g}.$$

One says that P is a $\mathfrak{g}$-*module.* A $\mathcal{K}$-multilinear skew-symmetric map

$$c^k:\overset{k}{\times}\mathfrak{g}\to P$$

is called a P-valued k-cochain on the Lie algebra $\mathfrak{g}$. These cochains form a $\mathfrak{g}$-module $C^k[\mathfrak{g};P]$. Let us put $C^0[\mathfrak{g};P]=P$. We obtain the cochain complex

$$0\to P\xrightarrow{\delta^0}C^1[\mathfrak{g};P]\xrightarrow{\delta^1}\cdots C^k[\mathfrak{g};P]\xrightarrow{\delta^k}\cdots, \tag{5.2.1}$$

with respect to the *Chevalley–Eilenberg coboundary operators*

$$\delta^k c^k(\varepsilon_0,\ldots,\varepsilon_k)=\sum_{i=0}^k(-1)^i\varepsilon_i c^k(\varepsilon_0,\ldots,\widehat{\varepsilon}_i,\ldots,\varepsilon_k)+ \tag{5.2.2}$$
$$\sum_{1\leq i<j\leq k}(-1)^{i+j}c^k([\varepsilon_i,\varepsilon_j],\varepsilon_0,\ldots,\widehat{\varepsilon}_i,\ldots,\widehat{\varepsilon}_j,\ldots,\varepsilon_k),$$

where the caret $\widehat{}$ denotes omission [39]. The complex (5.2.1) is called the *Chevalley–Eilenberg complex* with coefficients in a module P. It is finite if the Lie algebra $\mathfrak{g}$ is finite-dimensional.

For instance,

$$\delta^0 p(\varepsilon_0)=\varepsilon_0 p, \tag{5.2.3}$$
$$\delta^1 c^1(\varepsilon_0,\varepsilon_1)=\varepsilon_0 c^1(\varepsilon_1)-\varepsilon_1 c^1(\varepsilon_0)-c^1([\varepsilon_0,\varepsilon_1]). \tag{5.2.4}$$

Cohomology $H^*(\mathfrak{g};P)$ of the complex $C^*[\mathfrak{g};P]$ is called the *Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in the module P.*

The following are two standard variants of the Chevalley–Eilenberg complex.

(i) Let $P=\mathfrak{g}$ be regarded as a $\mathfrak{g}$-module with respect to the adjoint representation

$$\varepsilon:\varepsilon'\mapsto[\varepsilon,\varepsilon']\qquad,\varepsilon,\varepsilon'\in\mathfrak{g}.$$

We abbreviate with $C^*[\mathfrak{g}]$ the Chevalley–Eilenberg complex of $\mathfrak{g}$-valued cochains on $\mathfrak{g}$. Cohomology $H^*(\mathfrak{g})$ of this complex is called the *Chevalley–Eilenberg cohomology* or, simply, the *cohomology of a Lie algebra* $\mathfrak{g}$.

In particular, $C^0[\mathfrak{g}] = \mathfrak{g}$, while $C^1[\mathfrak{g}]$ consists of endomorphisms of the Lie algebra $\mathfrak{g}$. Accordingly, the coboundary operators (5.2.3) and (5.2.4) read

$$\delta^0\varepsilon(\varepsilon_0) = [\varepsilon_0, \varepsilon], \tag{5.2.5}$$
$$\delta^1 c^1(\varepsilon_0, \varepsilon_1) = [\varepsilon_0, c^1(\varepsilon_1)] - [\varepsilon_1, c^1(\varepsilon_0)] - c^1([\varepsilon_0, \varepsilon_1]). \tag{5.2.6}$$

A glance at the expression (5.2.6) shows that a one-cocycle c^1 on $\mathfrak{g}$ obeys the relation

$$c^1([\varepsilon_0, \varepsilon_1]) = [c^1(\varepsilon_0), \varepsilon_1] + [\varepsilon_0, c^1(\varepsilon_1)]$$

and, thus, it is a derivation of the Lie algebra $\mathfrak{g}$. Accordingly, any one-coboundary (5.2.5) is an inner derivation of $\mathfrak{g}$ up to the sign minus. Therefore, one can think of the cohomology $H^1(\mathfrak{g})$ as being the set of outer derivations of $\mathfrak{g}$.

(ii) Let $P = \mathcal{K}$ and $\mathfrak{g} : \mathcal{K} \to 0$. Then the Chevalley–Eilenberg complex $C^*[\mathfrak{g}; \mathcal{K}]$ is the exterior algebra $\wedge\mathfrak{g}^*$ of the dual Lie algebra $\mathfrak{g}^*$. The Chevalley–Eilenberg coboundary operators (5.2.2) on this algebra read

$$\delta^k c^k(\varepsilon_0, \ldots, \varepsilon_k) = \sum_{i<j}^{k} (-1)^{i+j} c^k([\varepsilon_i, \varepsilon_j], \varepsilon_0, \ldots, \widehat{\varepsilon}_i, \ldots, \widehat{\varepsilon}_j, \ldots, \varepsilon_k). \tag{5.2.7}$$

In particular,

$$\begin{aligned}
&\delta^0 c^0(\varepsilon_0) = 0, \qquad c^0 \in \mathcal{K}, \\
&\delta^1 c^1(\varepsilon_0, \varepsilon_1) = -c^1([\varepsilon_0, \varepsilon_1]), \qquad c^1 \in \mathfrak{g}^*, \\
&\delta^2 c^2(\varepsilon_0, \varepsilon_1, \varepsilon_2) = -c^2([\varepsilon_0, \varepsilon_1], \varepsilon_2) + c^2([\varepsilon_0, \varepsilon_2], \varepsilon_1) - c^2([\varepsilon_1, \varepsilon_2], \varepsilon_0).
\end{aligned} \tag{5.2.8}$$

Cohomology $H^*(\mathfrak{g}; \mathcal{K})$ of the complex $C^*[\mathfrak{g}; \mathcal{K}]$ are called the *Chevalley–Eilenberg cohomology with coefficients in the trivial representation*. It is provided with the *cup-product*

$$[c] \smile [c'] = [c \wedge c'], \tag{5.2.9}$$

where $[c]$ denotes the cohomology class of a cocycle c. This product makes $H^*(\mathfrak{g}; \mathcal{K})$ into a graded commutative algebra.

For instance, let $\mathfrak{g}$ be the right Lie algebra of a finite-dimensional real Lie group G. Then the relation (5.2.8) is the well-known Maurer–Cartan equation. Written with respect to the basis ε_i for $\mathfrak{g}$ and the dual basis θ^i for $\mathfrak{g}^*$, this equation reads

$$\delta\theta^k = -\frac{1}{2} c^k_{ij} \theta^i \wedge \theta^j. \tag{5.2.10}$$

There is a monomorphism of the complex $C^*[\mathfrak{g}; \mathbb{R}]$ onto the subcomplex of right-invariant exterior forms of the de Rham complex (1.6.4) of exterior forms on G. This monomorphism induces an isomorphism of the Chevalley–Eilenberg cohomology $H^*(\mathfrak{g}; \mathbb{R})$ to the de Rham cohomology of G [39]. For instance, if G is semi-simple, then

$$H^1(\mathfrak{g}; \mathbb{R}) = H^2(\mathfrak{g}; \mathbb{R}) = 0.$$

5.3 Sheaf cohomology

In this Section, we follow the terminology of [11, 47].

A *sheaf* on a topological space X is a topological fibre bundle $\pi : S \to X$ in modules over a commutative ring $\mathcal{K}$, where the surjection π is a local homeomorphism and fibres S_x, $x \in X$, called the *stalks*, are provided with the discrete topology. Global sections of a sheaf S make up a $\mathcal{K}$-module $S(X)$, called the *structure module* of S.

Any sheaf is generated by a presheaf. A *presheaf* $S_{\{U\}}$ on a topological space X is defined if a module S_U over a commutative ring $\mathcal{K}$ is assigned to every open subset $U \subset X$ ($S_\emptyset = 0$) and if, for any pair of open subsets $V \subset U$, there exists the *restriction morphism* $r^U_V : S_U \to S_V$ such that

$$r^U_U = \mathrm{Id}\, S_U, \qquad r^U_W = r^V_W r^U_V, \qquad W \subset V \subset U.$$

Every presheaf $S_{\{U\}}$ on a topological space X yields a sheaf on X whose stalk S_x at a point $x \in X$ is the direct limit of the modules S_U, $x \in U$, with respect to the restriction morphisms r^U_V. It means that, for each open neighborhood U of a point x, every element $s \in S_U$ determines an element $s_x \in S_x$, called the *germ* of s at x. Two elements $s \in S_U$ and $s' \in S_V$ belong to the same germ at x iff there exists an open neighborhood $W \subset U \cap V$ of x such that $r^U_W s = r^V_W s'$.

Example 5.3.1: Let $C^0_{\{U\}}$ be the presheaf of continuous real functions on a topological space X. Two such functions s and s' define the same germ s_x if they coincide on an open neighborhood of x. Hence, we obtain the *sheaf* C^0_X *of continuous functions* on X. Similarly, the *sheaf* C^∞_X *of smooth functions* on a smooth manifold X is defined. Let us also mention the presheaf of real functions which are constant on connected open subsets of X. It generates the *constant sheaf* on X denoted by $\mathbb{R}$. $\diamondsuit$

Two different presheaves may generate the same sheaf. Conversely, every sheaf S defines a presheaf $S(\{U\})$ of modules $S(U)$ of its local sections. It is called the *canonical presheaf* of the sheaf S. Global sections of S make up the *structure module* $S(X)$ of S. If a sheaf S is constructed from a presheaf $S_{\{U\}}$, there are natural module morphisms

$$S_U \ni s \mapsto s(U) \in S(U), \qquad s(x) = s_x, \quad x \in U,$$

which are neither monomorphisms nor epimorphisms in general. For instance, it may happen that a non-zero presheaf defines a zero sheaf. The sheaf generated by the canonical presheaf of a sheaf S coincides with S.

A direct sum and a tensor product of presheaves (as families of modules) and sheaves (as fibre bundles in modules) are naturally defined. By virtue of Theorem 1.1.3, a direct sum (resp. a tensor product) of presheaves generates a direct sum (resp. a tensor product) of the corresponding sheaves.

Remark 5.3.2: In the terminology of [84], a sheaf is introduced as a presheaf which satisfies the following additional axioms.

(S1) Suppose that $U \subset X$ is an open subset and $\{U_\alpha\}$ is its open cover. If $s, s' \in S_U$ obey the condition

$$r^U_{U_\alpha}(s) = r^U_{U_\alpha}(s')$$

for all U_α, then $s = s'$.

(S2) Let U and $\{U_\alpha\}$ be as in previous item. Suppose that we are given a family of presheaf elements $\{s_\alpha \in S_{U_\alpha}\}$ such that

$$r^{U_\alpha}_{U_\alpha \cap U_\lambda}(s_\alpha) = r^{U_\lambda}_{U_\alpha \cap U_\lambda}(s_\lambda)$$

for all U_α, U_λ. Then there exists a presheaf element $s \in S_U$ such that $s_\alpha = r^U_{U_\alpha}(s)$.

Canonical presheaves are in one-to-one correspondence with presheaves obeying these axioms. For instance, the presheaves of continuous, smooth and locally constant functions in Example 5.3.1 satisfy the axioms (S1) – (S2). $\diamondsuit$

Remark 5.3.3: The notion of a sheaf can be extended to sets, but not to non-commutative groups. One can consider a presheaf of such groups, but it generates a sheaf of sets because a direct limit of non-commutative groups need not be a group. $\diamondsuit$

There is a useful construction of a sheaf on a topological space X from local sheaves on open subsets which make up a cover of X.

PROPOSITION 5.3.1: Let $\{U_\zeta\}$ be an open cover of a topological space X and S_ζ a sheaf on U_ζ for every U_ζ. Let us suppose that, if $U_\zeta \cap U_\xi \neq \emptyset$, there is a sheaf isomorphism

$$\rho_{\zeta\xi} : S_\xi \mid_{U_\zeta \cap U_\xi} \to S_\zeta \mid_{U_\zeta \cap U_\xi}$$

and, for every triple $(U_\zeta, U_\xi, U_\iota)$, these isomorphisms fulfil the cocycle condition

$$\rho_{\xi\zeta} \circ \rho_{\zeta\iota}(S_\iota|_{U_\zeta \cap U_\xi \cap U_\iota}) = \rho_{\xi\iota}(S_\iota|_{U_\zeta \cap U_\xi \cap U_\iota}).$$

Then there exists a sheaf S on X together with the sheaf isomorphisms $\phi_\zeta : S|_{U_\zeta} \to S_\zeta$ such that

$$\phi_\zeta|_{U_\zeta \cap U_\xi} = \rho_{\zeta\xi} \circ \phi_\xi|_{U_\zeta \cap U_\xi}.$$

$\Box$

A *morphism of a presheaf* $S_{\{U\}}$ to a presheaf $S'_{\{U\}}$ on the same topological space X is defined as a set of module morphisms $\gamma_U : S_U \to S'_U$ which commute with restriction morphisms. A morphism of presheaves yields a *morphism of sheaves* generated by these presheaves. This is a bundle morphism over X such that $\gamma_x : S_x \to S'_x$ is the direct limit of morphisms γ_U, $x \in U$. Conversely, any morphism of sheaves $S \to S'$ on a topological space X yields a morphism of canonical presheaves of local sections of these sheaves. Let $\mathrm{Hom}\,(S|_U, S'|_U)$ be the commutative group of sheaf morphisms $S|_U \to S'|_U$ for any

open subset $U \subset X$. These groups are assembled into a presheaf, and define the sheaf $\mathrm{Hom}\,(S,S')$ on X. There is a monomorphism

$$\mathrm{Hom}\,(S,S')(U) \to \mathrm{Hom}\,(S(U),S'(U)), \qquad (5.3.1)$$

which need not be an isomorphism.

By virtue of Theorem 1.1.4, if a presheaf morphism is a monomorphism or an epimorphism, so is the corresponding sheaf morphism. Furthermore, the following holds.

THEOREM 5.3.2: A short exact sequence

$$0 \to S'_{\{U\}} \to S_{\{U\}} \to S''_{\{U\}} \to 0 \qquad (5.3.2)$$

of presheaves on the same topological space yields the short exact sequence of sheaves generated by these presheaves

$$0 \to S' \to S \to S'' \to 0, \qquad (5.3.3)$$

where the *factor sheaf* $S'' = S/S'$ is isomorphic to that generated by the factor presheaf

$$S''_{\{U\}} = S_{\{U\}}/S'_{\{U\}}.$$

If the exact sequence of presheaves (5.3.2) is split, i.e.,

$$S_{\{U\}} \cong S'_{\{U\}} \oplus S''_{\{U\}},$$

the corresponding splitting

$$S \cong S' \oplus S''$$

of the exact sequence of sheaves (5.3.3) holds. □

The converse is more intricate. A sheaf morphism induces a morphism of the corresponding canonical presheaves. If $S \to S'$ is a monomorphism, $S(\{U\}) \to S'(\{U\})$ is also a monomorphism. However, if $S \to S'$ is an epimorphism, $S(\{U\}) \to S'(\{U\})$ need not be so. Therefore, the short exact sequence (5.3.3) of sheaves yields the exact sequence of the canonical presheaves

$$0 \to S'(\{U\}) \to S(\{U\}) \to S''(\{U\}), \qquad (5.3.4)$$

where $S(\{U\}) \to S''(\{U\})$ is not necessarily an epimorphism. At the same time, there is the short exact sequence of presheaves

$$0 \to S'(\{U\}) \to S(\{U\}) \to S''_{\{U\}} \to 0, \qquad (5.3.5)$$

where the factor presheaf

$$S''_{\{U\}} = S(\{U\})/S'(\{U\})$$

generates the factor sheaf $S'' = S/S'$, but need not be its canonical presheaf.

THEOREM 5.3.3: Let the exact sequence of sheaves (5.3.3) be split. Then

$$S(\{U\}) \cong S'(\{U\}) \oplus S''(\{U\}),$$

and the canonical presheaves make up the short exact sequence

$$0 \to S'(\{U\}) \to S(\{U\}) \to S''(\{U\}) \to 0. \tag{5.3.6}$$

□

Let us turn now to sheaf cohomology. We follow its definition in [47]. In the case of paracompact topological spaces, it coincides with a different definition of sheaf cohomology based on the canonical flabby resolution (see Remark 5.3.6 below).

Let $S_{\{U\}}$ be a presheaf of modules on a topological space X, and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of X. One constructs a cochain complex where a p-cochain is defined as a function s^p which associates an element

$$s^p(i_0,\ldots,i_p) \in S_{U_{i_0}\cap\cdots\cap U_{i_p}}$$

to each $(p+1)$-tuple $(i_0,\ldots,i_p)$ of indices in I. These p-cochains are assembled into a module $C^p(\mathfrak{U}, S_{\{U\}})$. Let us introduce the coboundary operator

$$\begin{aligned}
&\delta^p : C^p(\mathfrak{U}, S_{\{U\}}) \to C^{p+1}(\mathfrak{U}, S_{\{U\}}),\\
&\delta^p s^p(i_0,\ldots,i_{p+1}) = \sum_{k=0}^{p+1}(-1)^k r_W^{W_k} s^p(i_0,\ldots,\widehat{i}_k,\ldots,i_{p+1}),\\
&W = U_{i_0}\cap\ldots\cap U_{i_{p+1}}, \qquad W_k = U_{i_0}\cap\cdots\cap \widehat{U}_{i_k}\cap\cdots\cap U_{i_{p+1}}.
\end{aligned} \tag{5.3.7}$$

One can easily check that $\delta^{p+1}\circ\delta^p = 0$. Thus, we obtain the cochain complex of modules

$$0 \to C^0(\mathfrak{U}, S_{\{U\}}) \xrightarrow{\delta^0} \cdots C^p(\mathfrak{U}, S_{\{U\}}) \xrightarrow{\delta^p} C^{p+1}(\mathfrak{U}, S_{\{U\}}) \longrightarrow \cdots. \tag{5.3.8}$$

Its cohomology groups

$$H^p(\mathfrak{U}; S_{\{U\}}) = \mathrm{Ker}\,\delta^p/\mathrm{Im}\,\delta^{p-1}$$

are modules. Of course, they depend on an open cover $\mathfrak{U}$ of the topological space X.

Remark 5.3.4: Throughout the Lectures, only *proper covers* are considered, i.e., $U_i \neq U_j$ if $i \neq j$. A cover $\mathfrak{U}'$ is said to be a *refinement* of a cover $\mathfrak{U}$ if, for each $U' \in \mathfrak{U}'$, there exists $U \in \mathfrak{U}$ such that $U' \subset U$. ♢

Let $\mathfrak{U}'$ be a refinement of the cover $\mathfrak{U}$. Then there is a morphism of cohomology groups

$$H^*(\mathfrak{U}; S_{\{U\}}) \to H^*(\mathfrak{U}'; S_{\{U\}}).$$

Let us take the direct limit of cohomology groups $H^*(\mathfrak{U}; S_{\{U\}})$ with respect to these morphisms, where $\mathfrak{U}$ runs through all open covers of X. This limit $H^*(X; S_{\{U\}})$ is called the *cohomology of* X *with coefficients in the presheaf* $S_{\{U\}}$.

Let S be a sheaf on a topological space X. *Cohomology of* X *with coefficients in* S or, simply, *sheaf cohomology* of X is defined as cohomology

$$H^*(X; S) = H^*(X; S(\{U\}))$$

with coefficients in the canonical presheaf $S(\{U\})$ of the sheaf S.

In this case, a p-cochain $s^p \in C^p(\mathfrak{U}, S(\{U\}))$ is a collection

$$s^p = \{s^p(i_0, \ldots, i_p)\}$$

of local sections $s^p(i_0, \ldots, i_p)$ of the sheaf S over $U_{i_0} \cap \cdots \cap U_{i_p}$ for each $(p+1)$-tuple $(U_{i_0}, \ldots, U_{i_p})$ of elements of the cover $\mathfrak{U}$. The coboundary operator (5.3.7) reads

$$\delta^p s^p(i_0, \ldots, i_{p+1}) = \sum_{k=0}^{p+1} (-1)^k s^p(i_0, \ldots, \widehat{i}_k, \ldots, i_{p+1})|_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}}. \tag{5.3.9}$$

For instance,

$$\delta^0 s^0(i, j) = [s^0(j) - s^0(i)]|_{U_i \cap U_j}, \tag{5.3.10}$$

$$\delta^1 s^1(i, j, k) = [s^1(j, k) - s^1(i, k) + s^1(i, j)]|_{U_i \cap U_j \cap U_k}. \tag{5.3.11}$$

A glance at the expression (5.3.10) shows that a zero-cocycle is a collection $s = \{s(i)\}_I$ of local sections of the sheaf S over $U_i \in \mathfrak{U}$ such that $s(i) = s(j)$ on $U_i \cap U_j$. It follows from the axiom (S2) in Remark 5.3.2 that s is a global section of the sheaf S, while each $s(i)$ is its restriction $s|_{U_i}$ to U_i. Consequently, the cohomology group $H^0(\mathfrak{U}, S(\{U\}))$ is isomorphic to the structure module $S(X)$ of global sections of the sheaf S. A one-cocycle is a collection $\{s(i, j)\}$ of local sections of the sheaf S over overlaps $U_i \cap U_j$ which satisfy the *cocycle condition*

$$[s(j, k) - s(i, k) + s(i, j)]|_{U_i \cap U_j \cap U_k} = 0. \tag{5.3.12}$$

If X is a paracompact space, the study of its sheaf cohomology is essentially simplified due to the following fact [47].

THEOREM 5.3.4: Cohomology of a paracompact space X with coefficients in a sheaf S coincides with cohomology of X with coefficients in any presheaf generating the sheaf S. □

Remark 5.3.5: We follow the definition of a *paracompact topological space* in [47] as a Hausdorff space such that any its open cover admits a *locally finite* open refinement, i.e., any point has an open neighborhood which intersects only a finite number of elements of this refinement.

A topological space X is paracompact iff any cover $\{U_\xi\}$ of X admits a subordinate *partition of unity* $\{f_\xi\}$, i.e.:

(i) f_ξ are real positive continuous functions on X;

(ii) $\operatorname{supp} f_\xi \subset U_\xi$;

(iii) each point $x \in X$ has an open neighborhood which intersects only a finite number of the sets $\operatorname{supp} f_\xi$;

(iv) $\sum\limits_\xi f_\xi(x) = 1$ for all $x \in X$. $\diamondsuit$

The key point of the analysis of sheaf cohomology is that short exact sequences of presheaves and sheaves yield long exact sequences of sheaf cohomology groups.

Let $S_{\{U\}}$ and $S'_{\{U\}}$ be presheaves on the same topological space X. It is readily observed that, given an open cover $\mathfrak{U}$ of X, any morphism $S_{\{U\}} \to S'_{\{U\}}$ yields a cochain morphism of complexes

$$C^*(\mathfrak{U}, S_{\{U\}}) \to C^*(\mathfrak{U}, S'_{\{U\}})$$

and the corresponding morphism

$$H^*(\mathfrak{U}, S_{\{U\}}) \to H^*(\mathfrak{U}, S'_{\{U\}})$$

of cohomology groups of these complexes. Passing to the direct limit through all refinements of $\mathfrak{U}$, we come to a morphism of the cohomology groups

$$H^*(X, S_{\{U\}}) \to H^*(X, S'_{\{U\}})$$

of X with coefficients in the presheaves $S_{\{U\}}$ and $S'_{\{U\}}$. In particular, any sheaf morphism $S \to S'$ yields a morphism of canonical presheaves $S(\{U\}) \to S'(\{U\})$ and the corresponding cohomology morphism

$$H^*(X, S) \to H^*(X, S').$$

By virtue of Theorems 5.1.1 and 5.1.2, every short exact sequence

$$0 \to S'_{\{U\}} \longrightarrow S_{\{U\}} \longrightarrow S''_{\{U\}} \to 0 \tag{5.3.13}$$

of presheaves on the same topological space X and the corresponding exact sequence of complexes (5.3.8) yield the long exact sequence

$$\begin{aligned} 0 \to H^0(X; S'_{\{U\}}) &\longrightarrow H^0(X; S_{\{U\}}) \longrightarrow H^0(X; S''_{\{U\}}) \longrightarrow \\ H^1(X; S'_{\{U\}}) &\longrightarrow \cdots H^p(X; S'_{\{U\}}) \longrightarrow H^p(X; S_{\{U\}}) \longrightarrow \\ H^p(X; S''_{\{U\}}) &\longrightarrow H^{p+1}(X; S'_{\{U\}}) \longrightarrow \cdots \end{aligned} \tag{5.3.14}$$

of the cohomology groups of X with coefficients in these presheaves. This result however is not extended to an exact sequence of sheaves, unless X is a paracompact space. Let

$$0 \to S' \longrightarrow S \longrightarrow S'' \to 0 \tag{5.3.15}$$

be a short exact sequence of sheaves on X. It yields the short exact sequence of presheaves (5.3.5) where the presheaf $S''_{\{U\}}$ generates the sheaf S''. If X is paracompact,

$$H^*(X; S''_{\{U\}}) = H^*(X; S'')$$

in accordance with Theorem 5.3.4, and we have the exact sequence of sheaf cohomology

$$\begin{aligned} 0 \to H^0(X; S') &\longrightarrow H^0(X; S) \longrightarrow H^0(X; S'') \longrightarrow \\ &H^1(X; S') \longrightarrow \cdots H^p(X; S') \longrightarrow H^p(X; S) \longrightarrow \\ &H^p(X; S'') \longrightarrow H^{p+1}(X; S') \longrightarrow \cdots . \end{aligned} \tag{5.3.16}$$

Let us point out the following isomorphism between sheaf cohomology and singular (Čhech and Alexandery) cohomology of a paracompact space [11, 82].

THEOREM 5.3.5: The sheaf cohomology $H^*(X; \mathbb{Z})$ (resp. $H^*(X; \mathbb{Q})$, $H^*(X; \mathbb{R})$) of a paracompact topological space X with coefficients in the constant sheaf $\mathbb{Z}$ (resp. $\mathbb{Q}$, $\mathbb{R}$) is isomorphic to the singular cohomology of X with coefficients in the ring $\mathbb{Z}$ (resp. $\mathbb{Q}$, $\mathbb{R}$). □

Since singular cohomology is a *topological invariant* (i.e., homotopic topological spaces have the same singular cohomology) [82], the sheaf cohomology groups $H^*(X; \mathbb{Z})$, $H^*(X; \mathbb{Q})$, $H^*(X; \mathbb{R})$ of a paracompact space are also topological invariants.

Let us turn now to the abstract de Rham theorem which provides a powerful tool of studying algebraic systems on paracompact spaces.

Let us consider an exact sequence of sheaves

$$0 \to S \xrightarrow{h} S_0 \xrightarrow{h^0} S_1 \xrightarrow{h^1} \cdots S_p \xrightarrow{h^p} \cdots . \tag{5.3.17}$$

It is said to be a *resolution of the sheaf* S if each sheaf $S_{p\geq 0}$ is *acyclic*, i.e., its cohomology groups $H^{k>0}(X; S_p)$ vanish.

Any exact sequence of sheaves (5.3.17) yields the sequence of their structure modules

$$0 \to S(X) \xrightarrow{h_*} S_0(X) \xrightarrow{h^0_*} S_1(X) \xrightarrow{h^1_*} \cdots S_p(X) \xrightarrow{h^p_*} \cdots \tag{5.3.18}$$

which is always exact at terms $S(X)$ and $S_0(X)$ (see the exact sequence (5.3.4)). The sequence (5.3.18) is a cochain complex because $h^{p+1}_* \circ h^p_* = 0$. If X is a paracompact space and the exact sequence (5.3.17) is a resolution of S, the *abstract de Rham theorem* establishes an isomorphism of cohomology of the complex (5.3.18) to cohomology of X with coefficients in the sheaf S as follows [47].

THEOREM 5.3.6: Given a resolution (5.3.17) of a sheaf S on a paracompact topological space X and the induced complex (5.3.18), there are isomorphisms

$$H^0(X; S) = \operatorname{Ker} h^0_*, \qquad H^q(X; S) = \operatorname{Ker} h^q_* / \operatorname{Im} h^{q-1}_*, \qquad q > 0. \tag{5.3.19}$$

□

We will also refer to the following minor modification of Theorem 5.3.6 [40, 83].

THEOREM 5.3.7: Let

$$0 \to S \xrightarrow{h} S_0 \xrightarrow{h^0} S_1 \xrightarrow{h^1} \cdots \xrightarrow{h^{p-1}} S_p \xrightarrow{h^p} S_{p+1}, \qquad p > 1, \tag{5.3.20}$$

be an exact sequence of sheaves on a paracompact topological space X, where the sheaves S_q, $0 \leq q < p$, are acyclic, and let

$$0 \to S(X) \xrightarrow{h_*} S_0(X) \xrightarrow{h^0_*} S_1(X) \xrightarrow{h^1_*} \cdots \xrightarrow{h^{p-1}_*} S_p(X) \xrightarrow{h^p_*} S_{p+1}(X) \tag{5.3.21}$$

be the corresponding cochain complex of structure modules of these sheaves. Then the isomorphisms (5.3.19) hold for $0 \leq q \leq p$. □

Any sheaf on a topological space admits the canonical resolution by flabby sheaves as follows.

A sheaf S on a topological space X is called *flabby* (or *flasque* in the terminology of [84]), if the restriction morphism $S(X) \to S(U)$ is an epimorphism for any open $U \subset X$, i.e., if any local section of the sheaf S can be extended to a global section. A flabby sheaf is acyclic. Indeed, given an arbitrary cover $\mathfrak{U}$ of X, let us consider the complex $C^*(\mathfrak{U}, S(\{U\}))$ (5.3.8) for its canonical presheaf $S(\{U\})$. Since S is flabby, one can define a morphism

$$\begin{aligned} & h : C^p(\mathfrak{U}, S(\{U\})) \to C^{p-1}(\mathfrak{U}, S(\{U\})), \qquad p > 0, \\ & hs^p(i_0, \ldots, i_{p-1}) = j^* s^p(i_0, \ldots, i_{p-1}, j), \end{aligned} \tag{5.3.22}$$

where U_j is a fixed element of the cover $\mathfrak{U}$ and $j^* s^p$ is an extension of $s^p(i_0, \ldots, i_{p-1}, j)$ onto $U_{i_0} \cap \cdots \cap U_{i_{p-1}}$. A direct verification shows that h (5.3.22) is a homotopy operator for the complex $C^*(\mathfrak{U}, S(\{U\}))$ and, consequently, $H^{p>0}(\mathfrak{U}, S(\{U\})) = 0$.

Given an arbitrary sheaf S on a topological space X, let $S^0_F(\{U\})$ denote the presheaf of all (not-necessarily continuous) sections of the sheaf S. It generates a sheaf S^0_F on X, and coincides with the canonical presheaf of this sheaf. There are the natural monomorphisms $S(\{U\}) \to S^0_F(\{U\})$ and $S \to S^0_F$. It is readily observed that the sheaf S^0_F is flabby. Let us take the quotient S^0_F/S and construct the flabby sheaf

$$S^1_F = (S^0_F/S)^0_F.$$

Continuing the procedure, we obtain the exact sequence of sheaves

$$0 \to S \longrightarrow S^0_F \longrightarrow S^1_F \longrightarrow \cdots, \tag{5.3.23}$$

which is a resolution of S since all sheaves are flabby and, consequently, acyclic. It is called the *canonical flabby resolution* of the sheaf S. The exact sequence of sheaves (5.3.23) yields the complex of structure modules of these sheaves

$$0 \to S(X) \longrightarrow S^0_F(X) \longrightarrow S^1_F(X) \longrightarrow \cdots . \tag{5.3.24}$$

If X is paracompact, the cohomology of X with coefficients in the sheaf S coincides with cohomology of the complex (5.3.24) by virtue of Theorem 5.3.6.

Remark 5.3.6: An important peculiarity of flabby sheaves is that a short exact sequence of flabby sheaves on an arbitrary topological space provides the short exact sequence of their structure modules. Therefore, there is a different definition of sheaf cohomology. Cohomology of a topological space X with coefficients in a sheaf S is defined directly as cohomology of the complex (5.3.24) [11]. For a paracompact space, this definition coincides with above mentioned one due to Theorem 5.3.6. $\diamondsuit$

In the sequel, we also refer to a *fine resolution* of sheaves, i.e., a resolution by fine sheaves.

A sheaf S on a paracompact space X is called *fine* if, for each locally finite open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X, there exists a system $\{h_i\}$ of endomorphisms $h_i : S \to S$ such that:

(i) there is a closed subset $V_i \subset U_i$ and $h_i(S_x) = 0$ if $x \notin V_i$,

(ii) $\sum\limits_{i\in I} h_i$ is the identity map of S. A fine sheaf on a paracompact space is acyclic.

Indeed, given an arbitrary locally finite cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X and a p-cochain s^p, let us define the $(p-1)$-cochain

$$\mathbf{h}s^p(i_0,\ldots,i_{p-1}) = \sum_{i\in I} i^*s^p(i,i_0,\ldots,i_{p-1}) \tag{5.3.25}$$

where i^*s^p, by definition, is equal to h_is^p on the set $U_i \cap U_{i_0} \cap \cdots U_{i_{p-1}}$ and to 0 outside this set. Then the morphism $\mathbf{h}$ (5.3.25) is a homotopy operator.

There are the following important examples of fine sheaves [41].

PROPOSITION 5.3.8: Let X be a paracompact topological space which admits a partition of unity performed by elements of the structure module $\mathfrak{A}(X)$ of some sheaf $\mathfrak{A}$ of real functions on X. Then any sheaf S of $\mathfrak{A}$-modules on X, including $\mathfrak{A}$ itself, is fine. $\Box$

In particular, the sheaf C^0_X of continuous functions on a paracompact topological space is fine, and so is any sheaf of C^0_X-modules. A smooth manifold X admits a partition of unity performed by smooth real functions. It follows that the sheaf C^∞_X of smooth real functions on X is fine, and so is any sheaf of C^∞_X-modules, e.g., the sheaves of sections of smooth vector bundles over X.

We complete our exposition of sheaf cohomology with the following useful theorem [5].

THEOREM 5.3.9: Let $f : X \to X'$ be a continuous map and S a sheaf on X. Let either f be a closed immersion or every point $x' \in X'$ have a base of open neighborhoods $\{U\}$ such that the sheaves $S\mid_{f^{-1}(U)}$ are acyclic. Then the cohomology groups $H^*(X;S)$ and $H^*(X';f_*S)$ are isomorphic. $\Box$

Bibliography

[1] A.Almorox, Supergauge theories in graded manifolds, In: *Differential Geometric Methods in Mathematical Physics*, Lect. Notes in Math. **1251** (Springer, Berlin, 1987), p.114.

[2] J.Anandan and Y.Aharonov, Geometric quantum phase and angles, *Phys. Rev. D* **38** (1988) 1863.

[3] P.Aschieri and P.Schupp, Vector fields on quantum groups, *Int. J. Mod. Phys. A* **11** (1996) 1077.

[4] M.Asorey, J.Cariñena and M.Paramion, Quantum evolution as a parallel transport, *J. Math. Phys.* **23** (1982) 1451.

[5] C.Bartocci, U.Bruzzo and D.Hernández Ruipérez, *The Geometry of Supermanifolds* (Kluwer Academic Publ., Dordrecht, 1991).

[6] M.Batchelor, The structure of supermanifolds, *Trans. Amer. Math. Soc.* **253** (1979) 329.

[7] M.Batchelor, Two approaches to supermanifolds, *Trans. Amer. Math. Soc.* **258** (1980) 257.

[8] A.Bohm and A.Mostafazadeh, The relation between the Berry and the Anandan–Ahoronov connections for $U(N)$ bundles, *J. Math. Phys.* **35** (1994) 1463.

[9] A.Borowiec, Vector fields and differential operators: Noncommutative case, *Cech. J. Phys.* **47** (1997) 1093; *E-print arXiv*: q-alg/9710006.

[10] O.Bratteli and D.Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol.1, Second Edition* (Springer, Berlin, 2002).

[11] G.Bredon, *Sheaf theory* (McGraw-Hill, N.-Y., 1967).

[12] U.Bruzzo, Supermanifolds, supermanifold cohomology, and super vector bundles, In: *Differential Geometric Methods in Theoretical Physics* (Kluwer, Dordrecht, 1988), p. 417.

[13] U.Bruzzo and V.Pestov, On the structure of DeWitt supermanifolds, *J. Geom. Phys.* **30** (1999) 147.

[14] J.-L. Brylinski, *Loop spaces, Characteristic Classes and Geometric Quantization* (Birkhäuser, Boston, 1993).

[15] A.Carey, D.Crowley and M.Murray, Principal bundles and the Dixmier-Douady class, *Commun. Math. Phys.* **193** (1998) 171.

[16] P.S.Chakraborty and A.Pla, Spectral triples and associated Connes–de Rham complex for the quantum $SU(2)$ and the quantum sphere, *Commun. Math. Phys.* **240** (2003) 447.

[17] P.S.Chakraborty and K.B.Sinha, Geometry on the quantum Heisenberg manifold, *J. Funct. Anal.* **203** (2003) 425.

[18] Z.Chang, Quantum group and quantum symmetry, *Phys. Rep.* **262** (1995) 137.

[19] V.Chari and A.Prestly, *A Guide to Quantum Groups* (Cambridge Univ. Press., Cambridge, 1994).

[20] R.Cianci, *Introduction to Supermanifolds* (Bibliopolis, Naples, 1990).

[21] R.Cirelli, A.Manià and L.Pizzocchero, Quantum mechanics as an infinite-dimensional Hamiltonian system with uncertainty structure, *J. Math. Phys.* **31** (1990) 2891, 2898.

[22] A.Connes, C^*-algèbres et géométrie différentielle, *C. R. Acad. Sci. Paris, Série A* **290** (1980) 599; *E-print arXiv:* hep-th/0101093.

[23] A.Connes, *Noncommutative Geometry* (Academic Press, New York, 1994).

[24] A.Connes, Gravity coupled with matter and the foundation of non-commutative geometry, *Commun. Math. Phys.* **182** (1996) 155.

[25] A.Connes, A short survey of noncommutative geometry, *J. Math. Phys.* **41** (2000) 3832.

[26] A.Connes and M.Dubois-Violette, Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examles, *Commun. Math. Phys.* **230** (2002) 539.

[27] J.Cuntz and D.Quillen, Algebra extension and nonsingularity, *J. Amer. Math. Soc.* **8** (1995) 251.

[28] L.Dabrowski, P.Hajac, G.Landi and P.Siniscalco, Metrics and pairs of left and right connections on bimodules, *J. Math. Phys.* **37** (1996) 4635.

[29] Ju.Daleckiĭ and M.Kreĭn, *Stability of Solutions of Differential Equations in Banch Space*, Transl. Math. Monographs (AMS, Providence, 1974).

[30] B.DeWitt, *Supermanifolds* (Cambridge Univ. Press, Cambridge, 1984).

[31] J.Dixmier, *C^*-Algebras* (North-Holland, Amsterdam, 1977).

[32] M.Dubois-Violette, Dérivations et calcul différentiel non-commutatif, *C. R. Acad. Sci. Paris* **307**, I (1988) 403.

[33] M.Dubois-Violette, R.Kerner and J.Madore, Noncommutative differential geometry of matrix algebras, *J. Math. Phys.* **31** (1990) 316.

[34] M.Dubois-Violette and T.Masson, On the first-order operators on bimodules, *Lett. Math. Phys.* **37** (1996) 467.

[35] M.Dubois-Violette and P.Michor, Connections on central bimodules in noncommutative differential geometry, *J. Geom. Phys.* **20** (1996) 218.

[36] M.Dubois-Violette, J.Madore, T.Masson and J.Morad, On curvature in noncommutative geometry, *J. Math. Phys.* **37** (1996) 4089.

[37] M.Dubois-Violette, Lectures on graded differential algebras and noncommutative geometry, *Noncommutative Differential Geometry and Its Applications to Physics*, Y. Maeda et al (eds) (Kluwer, 2001) pp. 245-306.

[38] J.Fröhlich, O. Grandjean and A. Recknagel, Supersymmetric quantum theory and non-commutative geometry, *Commun. Math. Phys.* **203** (1999) 117-184.

[39] D.Fuks, *Cohomology of Infinite-Dimensional Lie Algebras* (Consultants Bureau, New York, 1986).

[40] G.Giachetta, L.Mangiarotti and G.Sardanashvily, Cohomology of the infinite-order jet space and the inverse problem, *J. Math. Phys.* **42** (2001) 4272.

[41] G.Giachetta, L.Mangiarotti and G.Sardanashvily, *Geometric and Algebraic Topological Methods in Quantum Mechanics* (World Scientific, Singapore, 2005).

[42] G.Giachetta, L.Mangiarotti and G.Sardanashvily, *Advanced Classical Field Theory* (World Scientific, Singapore, 2009).

[43] J.Gracia-Bondía, J.Várilly and H.Figueroa, *Elements of Noncommutative Geometry* (Birkhauser, Boston, 2001).

[44] A.Grothendieck, *Eléments de Géométrie Algébrique IV*, Publ. Math. **32** (IHES, Paris, 1967).

[45] P.de la Harpe, *Classical Banach-Lie Algebras and Banach-Lie Groups of Operators in Hilbert Space*, Lect. Notes in Math. **285** (Springer, Berlin, 1972).

[46] I.Heckenberger and S.Kolb, Differential calculus on quantum homogeneous spaces, *Lett. Math. Phys.* **63** (2003) 255.

[47] F.Hirzebruch, *Topological Methods in Algebraic Geometry* (Springer, Berlin, 1966).

[48] P.Jara and D.Llena, Lie bracket of vector fields in noncommutative geometry, *E-print arXiv*: math.RA/0306044.

[49] A.Klimyk and K.Schmüdgen, *Quantum Groups and their Representations*, Texts and Monographs in Physics (Springer, Berlin, 1997).

[50] J.Koszul, *Lectures on Fibre Bundles and Differential Geometry* (Tata University, Bombay, 1960).

[51] I.Krasil'shchik, V.Lychagin and A.Vinogradov, *Geometry of Jet Spaces and Non-linear Partial Differential Equations* (Gordon and Breach, Glasgow, 1985).

[52] N.Kuiper, Contractibility of the unitary group of a Hilbert space, *Topology* **3** (1965) 19.

[53] G.Landi, *An Introduction to Noncommutative Spaces and their Geometries*, Lect. Notes in Physics, New series m: Monographs, **51** (Springer, Berlin, 1997).

[54] S.Lang, *Algebra* (Addison–Wisley, New York, 1993).

[55] S.Lang, *Differential and Riemannian Manifolds*, Gradutate Texts in Math. **160** (Springer, New York, 1995).

[56] V.Lunts and A.Rosenberg, Differential operators on noncommutative rings, *Selecta Mathematica, New Series* **3** (1997) 335.

[57] S.Mac Lane, *Homology* (Springer, Berlin, 1967).

[58] J.Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications* (Cambridge Univ. Press, Cambridge, 1995).

[59] S.Majid, *A Quantum Group Primer* (Cambr. Univ. Press, Cambridge, 2002).

[60] L.Mangiarotti and G.Sardanashvily, *Gauge Mechanics* (World Scientific, Singapore, 1998).

[61] L.Mangiarotti and G.Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).

[62] W.Massey, *Homology and Cohomology Theory* (Marcel Dekker, Inc., New York, 1978).

[63] A.Mishchenko, C^*-algebras and K-theory, In: *Algebraic Topology (Aarhus, 1978)*, Lect. Notes in Math. **763** (Springer, Berlin, 1979).

[64] J.Mourad, Linear connections in noncommutative geometry, *Class. Quant. Grav.* **12** (1995) 965.

[65] A.Pietsch, *Nuclear Locally Convex Spaces* (Springer, Berlin, 1972).

[66] A.Rennie, Commutative geometries are spin manifolds, *Rev. Math. Phys.* **13** (2001) 409.

[67] A.Rennie, Smoothness and locality for nonunital spectral triples, *K-Theory* **28** (2003) 127.

[68] M.Rieffel, Induced representations of C^*-algebras, *Adv. Math.* **13** (1974) 176.

[69] M.Rieffel, Morita equivalence for C^*-algebras and W^*-algebras, *J. Pure Appl. Alg.* **5** (1974) 51.

[70] A.Robertson and W.Robertson, *Topological Vector Spaces* (Cambridge Univ. Press., Cambridge, 1973).

[71] A.Rogers, A global theory of supermanifolds, *J. Math. Phys.* **21** (1980) 1352.

[72] A.Rogers, Graded manifolds, supermanifolds and infinite-dimensional Grassmann algebras, *Commun. Math. Phys.* **105** (1986) 375.

[73] A.Rogers, *Supermanifolds: Theory and Applications* (World Scientific, Singapore, 2007).

[74] D.Ruipérez and J.Masqué, Global variational calculus on graded manifolds, *J. Math. Pures et Appl.* **63** (1984) 283; **64** (1985) 87.

[75] G.Sardanashvily, Classical and quantum mechanics with time-dependent parameters, *J. Math. Phys.* **41** (2000) 5245; E-print arXiv: quant-ph/0004005.

[76] G.Sardanashvily and G.Giachetta, What is geometry in quantum theory, *Int. J. Geom. Methods Mod. Phys.* **1** (2004) 1-22; E-print arXiv: hep-th/0401080.

[77] G.Sardanashvily, A dilemma of nonequivalent definitions of differential operators in noncommutative geometry, *E-print arXiv:* math/0702850.

[78] G.Sardanashvily, Quantum mechanics with respect to different reference frames, *J. Math. Phys.* **48** (2007) 082104; *E-print arXiv:* quant-ph/0703266.

[79] G.Sardanashvily, Fibre bundles, jet manifolds and Lagrangian theory. Lectures for theoreticians, *E-print arXiv:* 0908.1886.

[80] G.Sardanashvily, Lectures on supergeometry, *E-print arXiv:* 0910.0092.

[81] K.Schmüdgen and A.Schüller, Left covariant differential calculi on $SL_q(2)$ and $SL_q(3)$, *J. Geom. Phys.* **20** (1996) 87.

[82] E.Spanier, *Algebraic Topology* (McGraw-Hill, N.Y., 1966).

[83] F.Takens, A global version of the inverse problem of the calculus of variations, *J. Diff. Geom.* **14** (1979) 543.

[84] B.Tennison, *Sheaf Theory* (Cambridge Univ. Press, Cambridge, 1975).

[85] F.Trevers, *Topological Vector Spaces, Distributions and Kernels* (Academic Press, New York, 1967).

[86] I.Vaisman, *Cohomology and Differential Forms* (Marcel Dekker, Inc., New York, 1973).

[87] J.Várilly and J.Gracia-Bondia, Connes' noncommutative differential geometry and the Standard Model, *J. Geom. Phys.* **12** (1993) 223.

[88] S.Woronowicz, Compact matrix pseudogroups, *Commun. Math. Phys.* **111** (1987) 613.

[89] S.Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), *Commun. Math. Phys.* **122** (1989) 125.

[90] S.Woronowicz, C^*-algebras generated by unbounded elements, *Rev. Math. Phys.* **7** (1994) 481.

Index

C^*-algebra
defined by a continuous field of C^*-algebras, 41
elementary, 41
C^*-dynamic system, 46
C^*-module, 82
G-superfunction, 72
G-supermanifold, 73
G-supermanifold standard, 73
GH^∞-superfunctions, 71
G^∞-superfunctions, 71
G^∞-tangent bundle, 77
G^∞-vector bundle, 77
H^∞-superfunctions, 71
K-cycle, 97
$\mathcal{K}$-algebra, 6
p-coboundary, 111
p-cocycle, 111

Atiyah class, 27

adjoint action
of a Hopf algebra, 104
algebra, 5
$\mathbb{N}$-graded, 17
commutative, 17
$\mathbb{Z}_2$-graded, 53
unital, 5
algebra $\mathbb{Z}_2$-graded commutative, 53
antiholomorphic function on a Hilbert space, 36
antipode, 101

Banach manifold, 30
Banach vector bundle, 40
basis
for a graded manifold, 63
for a module, 7
Batchelor's theorem, 62
Berry connection, 49
bi-algebra, 100
bimodule, 6
covariant, 107
bimodule commutative, 6
bimodule graded, 54
body
of a graded manifold, 62
of a ringed space, 19
body map, 55

Cartan pair, 89
category **Vect**($\mathcal{B}$), 32
category **Bnh**, 32
center of an algebra, 81
characteristic vector bundle, 63
Chevalley–Eilenberg
coboundary operator, 114
cohomology, 114
with coefficients in a module, 114
complex, 114
Chevalley–Eilenberg cohomology
with coefficients in the trivial representation, 115
Chevalley–Eilenberg differential calculus, 18
minimal, 19
co-operation, 101
coaction, 105
coalgebra, 100

coboundary operator, 111
cochain, 111
cochain complex, 111
cochain morphism, 112
cocycle condition, 120
cohomology
  of a Lie algebra, 114
  with coefficients in a presheaf, 120
cohomology group, 111
comodule
  over a Hopf algebra, 105
  trivial, 105
comodule algebra, 105
complex, 111
  acyclic, 111
  exact, 111
complex vector field
  on a Hilbert manifold, 36
comultiplication, 100
connection
  left, 92
  non-commutative
    universal, 92
  on a bundle of C^*-algebras, 43
  on a graded commutative ring, 62
  on a Hilbert bundle, 44
  on a Hilbert manifold, 38
  on a module, 15, 16
  on a ring, 16
  on a sheaf, 26
  right, 92
connection on a Banach manifold, 33
connection on a graded module, 61
Connes' differential calculus, 98
convolution product, 103
cotangent bundle
  complex, 37
  antiholomorphic, 37
  holomorphic, 37
  of a Banach manifold, 32
counit, 100
covariant differential
  on a module, 16
cup-product, 17
  of the Chevalley–Eilenberg cohomology, 115
curvature of a connection
  on a module, 16
curvature of a connection on sheaves, 26
curvature of a graded connection, 66
curvature of a superconnection, 78

de Rham cohomology, 17
  of a ring, 19
de Rham cohomology of a manifold, 22
de Rham complex, 17
  of a ring, 19
  of sheaves, 22
de Rham complex of exterior forms, 22
de Rham theorem, 22
de Rham theorem abstract, 122
derivation
  inner, 12
  of a non-commutative algebra, 12
  with values in a module, 11
derivation graded, 64
derivation module, 11
derivative
  on a Banach space, 30
differentiable function
  between Banach spaces, 30
  on a Hilbert space, 35
differential
  on a Banach space, 30
differential bigraded algebra, 60
differential calculus, 17
  minimal, 18
differential calculus universal, 86
differential graded algebra, 17
differential operator
  in non-commutative geometry, 88, 90
  on a module, 10

direct image
  of a sheaf, 19
direct limit, 8
direct sequence, 8
direct sum
  of complexes, 112
  of modules, 6
direct system
  of modules, 8
directed set, 8
division algebra, 5
Dixmier–Douady class, 43
dual Hopf algebra, 104
Dubois–Violette connection, 93
  linear, 95
  real, 95
Dubois–Violette connection dual, 94

endomorphism
  of a Hilbert module, 83
    adjoint, 83
    compact, 83
evaluation morphism, 71
even element, 53
even morphism, 55
evolution operator, 48
exact sequence
  of modules, 7
  short, 7
exterior algebra, 9
  bigraded, 54
exterior form
  antiholomorphic
    on a Hilbert manifold, 37
  on a Hilbert manifold, 37
exterior form on a Banach manifold, 33

factor algebra, 5
factor complex, 112
factor module, 7
factor sheaf, 118
fibre bundle
  of C^*-algebras, 41
fibred manifold Banach, 32
field, 5
field of C^*-algebras, 41
FODC, 106
  bicovariant, 106
  left-covariant, 106
  right-covariant, 106
Fréchet ring, 21
Fubini–Studi metric, 40
fundamental form
  of a Hermitian metric on a Hilbert manifold, 38

general linear graded group, 56
generalized connection, 44
geometric space, 19
graded commutative ring, 55
  Banach, 55
graded connection, 66
graded de Rham cohomology, 68
graded derivation, 57
graded derivation module, 58
graded differential calculus, 60
graded differential operator, 57
graded envelope, 56
graded exterior differential, 68
graded exterior form, 67
graded exterior product, 54
graded function, 62
graded jet module, 58
graded manifold, 62
  simple, 63
graded module, 54
  dual, 55
  free, 54
graded vector field, 64
graded vector space, 54
grading automorphism, 53
Grassmann algebra, 55

Grothendieck's topology, 21
group function Hopf algebra, 102
group Hopf algebra, 102

Heisenberg equation, 46
Hermitian form
  on a module, 82
    invertible, 82
    positive, 82
Hermitian manifold
  infinite-dimensional, 38
Hermitian metric
  on a Hilbert manifold, 38
Hilbert bundle, 41
Hilbert manifold, 36
Hilbert module, 41, 82
holomorphic exterior form on a Hilbert manifold, 37
holomorphic function on a Hilbert space, 36
homogeneous element, 54
homotopy operator, 111
Hopf algebra, 101
  classical, 102
  cocommutative, 101
  dually paired, 103
  involutive, 101
  quasi-triangular, 101

ideal, 5
  maximal, 5
  of nilpotents, 55
  prime, 5
ideal proper, 5
inductive limit, 9
instantwise quantization, 45
interior product
  graded, 60
invariant vector field, 109
inverse image
  of a sheaf, 20
inverse limit, 9
inverse mapping theorem, 30
inverse sequence, 9

jet module, 12
jet sheaf, 25
juxtaposition rule, 17

Kähler form
  on a Hilbert manifold, 38
Kähler manifold
  infinite-dimensional, 38
Kähler metric
  on a Hilbert manifold, 38
Koszul connection, 16

Leibniz rule, 11, 16
Leibniz rule graded, 17
Leibniz rule non-commutative, 12
Lie derivative
  of a bigraded algebra, 60
Lie superalgebra, 56
local ring, 5
local-ringed space, 19
locally
  free module, 20
locally finite covering, 120

matrix geometry, 95
Maurer–Cartan form
  on a Hopf algebra, 108
metric connection
  on a Hilbert manifold, 39
module, 6
  dual, 7
  finitely generated, 7
  free, 7
  involutive, 82
  of finite rank, 7
  over a Hopf algebra, 104
  over a Lie algebra, 114
  projective, 7
morphism

of presheaves, 117
of ringed spaces, 20
of sheaves, 117

non-commutative gauge field, 100
non-commutative vector field, 89

odd element, 53
odd morphism, 55
operator of a parallel displacement, 47

paracompact space, 120
partition of unity, 121
positive element, 82
pre-Hilbert module, 82
preaheaf
canonical, 116
presheaf, 116
product
of G-supermanifolds, 75
product G-supermanifold, 75
projective Hilbert space, 39
proper cover, 119

quantum Lie algebra of a Hopf algebra, 109
quantum tangent space, 109

refinement, 119
resolution, 112
fine, 124
flabby, 124
of a sheaf, 122
restriction morphism, 116
ring, 5
ringed space, 19

Schrödinger equation, 47
second-countable topological space, 21
separable topological space, 21
Serre–Swan theorem, 23
sheaf, 116
acyclic, 122
constant, 116
fine, 124
flabby, 123
flasque, 123
locally free, 20
of constant rank, 20
of finite type, 20
of continuous functions, 116
of derivations, 20
of jets, 25
of modules, 20
of smooth functions, 116
sheaf cohomology, 120
smooth manifold, 21
smooth supermanifold, 72
soul map, 55
spectral triple, 97
spectral triple even, 97
spectral triple odd, 97
split (injection), 29
split (subspace), 29
splitting domain, 62
splitting of an exact sequence, 8
structure module
of a sheaf, 116
of a simple graded manifold, 63
of a vector bundle, 23
structure sheaf
of a graded manifold, 62
of a ringed space, 19
of a vector bundle, 22
structure sheaf of a supermanifold, 72
superbracket, 56
superconnection, 78
superform, 73
superfunction, 69
smooth, 71
supermatrix, 56
supermatrix even, 56
supermatrix odd, 56
supertangent bundle, 77
supervector bundle, 76

supervector field, 73
supervector space, 56

tangent bundle
antiholomorphic, 36
complexified , 36
holomorphic, 36
tangent bundle of a Banach manifold, 31
tangent space
antiholomorphic, 36
tangent space complex, 36
tangent space holomorphic, 36
tangent space to a Banach manifold, 31
tensor algebra, 9
tensor product
of algebras, 100
of complexes, 112
of graded modules, 54
of modules, 6
topological, 21
time-ordered exponential, 48
topological invariant, 122
torsion
of a non-commutative connection, 95
trivial extension of a sheaf, 20

underlying G^∞-supermanifold, 73
unital extension, 6
universal R matrix, 101
universal form, 86

vector field
on a Banach manifold, 32
vector space, 6

Yang–Baxter equation
quantum, 101